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DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

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DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

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BLOCK I: EXACT, HOMOGENEOUS AND LINEAR DIFFERENTIAL EQUATIONS

UNIT- I

Structure:

1.1 INTRODUCTION

A differential equation is an equation which involves derivatives. The following are some examples of differential equations.

1. $y' = \sin x$.
2. $y'' + 3y' + 2y = e^x$.
3. $(y'')^2 + (y')^3 + 3y = x^2$.
4. $y''' + 3(y'')^2 + y' = e^x$.
5. $y = xy'' + r\sqrt{[1 + (y')^2]}$.
6. $\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = z$.
7. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y$.

In a differential equation if there is a single independent variable and the derivatives are ordinary derivatives then it is called an **ordinary differential equation**.

Examples 1 to 5 are ordinary differential equations.

If there are two or more independent variables and the derivatives are partial derivatives then it is called a **partial differential equation**.

Examples 6 and 7 are partial differential equations.

The **order** of a differential equation is the order of the highest derivative appearing in it.

Examples 1 and 6 are of first order; and 2, 3, 5, 7 are of order two and 4 is of order 3.

The **degree** of the differential equation is the degree of the highest ordered derivative occurring in it when the differential coefficients are free from radicals and fractions.

In the above examples all except 3 and 5 are of degree one; examples 3 and 5 are of degree two.

1.2 EXACT DIFFERENTIAL EQUATIONS

Definition.

Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0. \quad ------------ (1.1)$$

Suppose there exists a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M \text{ and } \frac{\partial f}{\partial y} = N.$$

Then the differential equation takes the form

$$\left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy = 0 \text{ (i.e.,) } df = 0.$$

On integration the general solution is given by $f(x,y) = c$. In this case the expression $Mdx + Ndy$ is said to be an **exact differential** and (1.1) is called an **exact differential equation**.

1.3 CONDITIONS FOR EQUATION TO BE EXACT

Theorem 1

The differential equation $Mdx + Ndy = 0$ is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Proof:

Suppose the equation is exact.

Then there exists a function $f(x,y)$ such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Conversly:

Suppose $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ .... (1.2)

We have to construct a function $f(x,y)$ such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$.

Let $f = \int Mdx + g(y)$ .... (1.3)

where g is an arbitrary function of y.

By definition of f, we have $\frac{\partial f}{\partial x} = M$.

Hence the problem is to determine $g(y)$ in such a way that $\frac{\partial f}{\partial y} = N$.

(ie.,) $\frac{\partial}{\partial y}\left[\int Mdx + g(y)\right] = N$.

(ie.,) $\frac{\partial}{\partial y}\left(\int Mdx\right) + g'(y) = N$.

$\therefore g'(y) = N - \frac{\partial}{\partial y}\left(\int Mdx\right)$ .... (1.4).

If we prove that the right hand side of (1.4) is a function of y only then we can integrate (1.4) to obtain $g(y)$.

Now,

$$\frac{\partial}{\partial x}\left[N-\frac{\partial}{\partial y}(fMdx)\right] = \frac{\partial N}{\partial x}-\frac{\partial^2}{\partial x\partial y}\left(\int Mdx\right)$$
$$= \frac{\partial N}{\partial x}-\frac{\partial^2}{\partial y\partial x}\left(\int Mdx\right)$$
$$= \frac{\partial N}{\partial x}-\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\int Mdx\right)\right)$$
$$= \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$$
$$= 0. \quad by\ (1.2).$$

This shows that the right hand side of (1.4) is a function of y only.

Hence (1.4) $\Rightarrow g(y)=\int\left[N-\frac{\partial}{\partial y}\int Mdx\right]dy.$

Substituting this value of $g(y)$ in (1.3) we get the required function f.

Hence the equation is exact and the general solution is given by $f(x,y)=c$.

1.4 WORKING RULE FOR SOLVING IT

1. Verify whether the given equation $Mdx+Ndy=0$ is exact.
   (i.e.,) verify $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.
2. If exact, integrate M w.r.t x keeping y as constant.
3. Find out those terms in N which are free from x and integrate those terms w.r.t. y.
4. The sum of these two expressions equated to an arbitrary constant is the required general solution of the given exact equation.

Note:(Grouping Method:)

In certain cases the function $f(x,y)$ can be guessed by grouping the terms of the equation properly.

For example, the equation $(y+\sin y)dx+(x+x\cos y)dy=0$ can be grouped as $(ydx+xdx)+(\sin ydx+x\cos ydy)=0$.

(ie.,) $d(xy)+d(x\sin y)=0$

Hence the solution is $xy+x\sin y=c$.

1.5 PROBLEMS

Problem 1. Verify whether $e^y dx+(xe^y+2y)dy=0$ is exact, if so solve.

Solution:

Here $M=e^y$ and $N=xe^y+2y$ and $\frac{\partial M}{\partial y}=e^y=\frac{\partial N}{\partial x}$.

Hence the equation is exact.

The given equation can be grouped as

$$(e^y dx+xe^y dy)+2ydy=0$$

$d(xe^y)+d(y^2)=0.$

$xe^y+y^2=c.$

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Problem 2. Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0.$

Solution:

Here $M = x^2 - 4xy - 2y^2$ and $N = y^2 - 4xy - 2x^2$ and $\frac{\partial M}{\partial y} = -4x - 4y = \frac{\partial N}{\partial x}$.

Hence the equation is exact.

Now, $\int Mdx = \frac{1}{3}x^3 - 2x^2y - 2y^2x.$

In N, the term free from x is y^2 whose integral w.r.t y is $\frac{1}{3}y^3$.

Hence the complete solution is $\frac{1}{3}x^3 - 2x^2y - 2y^2x + \frac{1}{3}y^3 = c'$.

(ie.,) $x^3 + y^3 - 6xy(x + y) = 3c' = c.$

Problem 3. Solve $xdx + ydy - \left(\frac{xdy - ydx}{x^2+y^2}\right) = 0.$

Solution:

The equation can be written as

$$\left(x + \frac{y}{x^2+y^2}\right)dx + \left(y - \frac{x}{x^2+y^2}dy\right) = 0.$$

Here $M = x + \frac{y}{x^2+y^2})$ and $N = y - \frac{x}{x^2+y^2}$ and $\frac{\partial M}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x}$.

Hence the equation is exact.

Now, $\int Mdx = \frac{1}{2}x^2 + \tan^{-1}(x/y).$

In N the term free from x is y whose integral w.r.t y is $\frac{1}{2}y^2$.

Hence the complete solution is $\frac{1}{2}x^2 + \frac{1}{2}y^2 + 2\tan^{-1}(x/y) = c'$

(i.e.,) $x^2 + y^2 + 2\tan^{-1}(x/y) = c.$

Exercises.

1. Verify whether the following differential equations are exact.

(i) $ydx - xdy = 0.$

(ii) $(x^2 - y)dx + (y^2 - x)dy = 0.$

2. Show that the following equations are exact differential equations and solve them.

(i) $(x^2 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0.$

(ii) $(x^2 - ay)dx + (y^2 - ax)dy = 0.$

UNIT- II EQUATIONS OF THE FIRST ORDER, BUT OF HIGHER DEGREE

Structure:

2.1 Equations solvable for $\frac{dy}{dx}$.
2.2 Equations solvable for y.
2.3 Equations solvable for x.
2.4 Clairaut's form.
2.5 Equations that do not contain (i) x explicitly (ii) y explicitly.

2.1 Equations solvable for $\frac{dy}{dx}$

We shall denote $\frac{dy}{dx}$ hereafter by p.
Let the equation of the first order and of the n^{th} degree be

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots + P_n = 0, \ldots\ldots\ldots (2.1)$$

where $P_1, P_2, \ldots, P_n$ denote functions of x and y.

Suppose the first member of (1) can be resolved into factors of the first degree of the form

$$(p - R_1)(p - R_2)(p - R_3)\ldots(p - R_n).$$

Any relation between x and y which makes any of the these factors vanish is a solution of (1). Let the primitive of $p - R_1 = 0$, $p - R_2 = 0$, etc., be $\phi_1(x, y, c_1) = 0$, $\phi_2(x, y, c_2) = 0$, ..., $\phi_n(x, y, c_n) = 0$ respectively, where $c_1, c_2, \ldots, c_n$ are arbirary constants. Without any loss of generality, we can replace $c_1, c_2, \ldots c_n$ by c, where c is an arbitrary constant.
Hence the solution of (2.1) is $\phi_1(x, y, c).\phi_2(x, y, c)\ldots\phi_n(x, y, c) = 0$.

<u>Problems:</u>

Problem 1. Solve $x^2p^2 + 3xyp + 2y^2 = 0$.

Solution:

Solving for p, $p = -\frac{y}{x}$ or $p = -\frac{2y}{x}$.

$\frac{dy}{dx} = -\frac{y}{x}$ gives $xy = c$.

$\frac{dy}{dx} = -\frac{2y}{x}$ gives $yx^2 = c$.

The solution is $(xy - c)(yx^2 - c) = 0$.

Problem 2. Solve $p^2 + \left(x + y - \frac{2y}{x}\right)p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0$.

Solution:

Solving for p, $p = \frac{y}{x} - y$ or $p = \frac{y}{x} - x$.

$\frac{dy}{y} = \left(\frac{1}{x} - 1\right)dx$ or $\frac{dy}{dx} - \frac{x}{y} = -x$.

The first equation gives

$\log\frac{y}{x} = -x + \log c$

therefore, $y = cxe^{-x}$.

The second equation is linear in y. Hence the solution is $\frac{y}{x} = -x + c$

i.e., $y + x^2 - cx = 0$.

The general solution is $(y - cxe^{-x})(y + x^2 - cx) = 0$.

2.2 Equations solvable for y

If $f(x, y, z) = 0$ can be put in the form

$$y = F(x, y) \text{ -------------- } (2.2)$$

Differentiating with respect to x, $p = \phi(x, p, \frac{dp}{dx})$.

This being an equation in the two variables p and x, can be integrated by any of the foregoing methods. Hence we obtain,

$$\psi(x, p, c) = 0 \text{ -------------- } (2.3)$$

Eliminating p between (2.2) and (2.3), the solution is got.

2.3 Equations solvable for x

Let $f(x, y, z) = 0$ be in this case put in the form

$$x = F(y, p) \text{ ----------------- } (2.4)$$

Differentiating with respect to y, $\frac{1}{p} = \phi(y, p, \frac{dp}{dy})$.

Integrating leads to $\psi(y, p, c) = 0$ --------------------- (2.5)

Eliminating p between (2.4) and (2.5), the solution is got.

Examples:

Example 1. Solve $xp^2 - 2yp + x = 0$.

Solution:

Given the equation is $xp^2 - 2yp + x = 0$.

Solving for y, $y = \frac{x(p^2+1)}{2p}$.

Differentiating with respect to x, $\frac{p^2-1}{p} = \frac{dp}{dx}.\frac{p^2-1}{p^2}x$.

$\therefore \frac{dx}{x} = \frac{dp}{p}$.

Integrating $p = cx$.

Eliminating p between this and the given equation, the solution is $2cy = c^2x^2 + 1$.

Example 2. Solve $x = y^2 + \log p$

Solution:

Given the equation is $x = y^2 + \log p$ ------------------- (2.6)

Differentiating with respect to y, $\frac{1}{p} = 2y + \frac{1}{p}\frac{dp}{dy}$. $\frac{dp}{dy} + 2py = 1$.

This is linear in p and hence $pe^{y^2} = \int e^{y^2}dy + c$ ----------------- (2.7)

The eliminate of p between (2.6) and (2.7) gives the solution.

2.4 Clairaut's form.

The equation known as Clairaut's is of the form

$$y = px = f(p) \text{ ------------ } (2.8)$$

Differentiating with respect to x,

$p = p + \{x + f'(p)\}\frac{dp}{dx}$. i.e., $\{x + f'(p)\}\frac{dp}{dx}$

Either $\frac{dp}{dx} = 0$ or $x + f'(p) = 0$.

$\frac{dp}{dx} = 0$ gives $p = c$, a constant.

The solution of (1) is $y = cx + f(c)$ -------------- (2.9)

We have to replace p in Clairaut's equation by c. The other factor $x + f'(p) = 0$ taken along with (2.8) give, on elimination of p, a solution of (2.8). But this solution is not included in the general solution of (2.9). Such a solution as this is called a singular solution.

Examples:

Example 1. Solve $y = (x - a)p - p^2$.

Solution:

The given equation is $y = (x - a)p - p^2$.

This is Clairaut's equation; hence the solution is

$$y = (x - a)c - c^2.$$

Example 2. Solve $y = 2px + y^2p^3$

Solution:

The given equation is $y = 2px + y^2p^3$

Putting $X = 2x$ and $Y = y^2$, the equation transforms into

$Y = XP + P^3$, where $P = \frac{dY}{dX} = py$.

This is Clairaut's equation; hence $Y = cX + c^3$.

The solution is $y^2 = 2xc + c^3$.

2.5 Equations that do not contain (i) x explicitly (ii) y explicitly

(i) Equations that do not contain x explicitly

Let the equation is of the form $f(y, p) = 0$ --------------- (2.10)

If this is solvable for p, then $p = \phi(y)$ and hence is immediately integrable.

If (2.10) is solvable for y, so that $y = \phi(p)$, then the equation solvable for y method is applied.

(ii) Equations that do not contain x explicitly

Let the equation is of the form $f(x, p) = 0$ -------------- (2.11)

If this is solvable for p, then $p = \phi(x)$, it is directly integrable.

If (1) is solvable for x, so that $y = \phi(p)$, then the equation solvable for x method is applied.

Examples:

Example 1: Solve $x^2 = (1 + p^2)$.

Solution:

Given the equation is $x^2 = (1 + p^2)$.

$x = \pm\sqrt{1 + p^2}$. Here y is explicitly absent.

Differentiate with respect to y

$$\frac{1}{p} = \frac{p}{\sqrt{(1+p^2)}}\frac{dp}{du}$$

$$\therefore dy = \frac{p^2}{\sqrt{(1+p^2)}}dp. \text{ ---------------- (2.12)}$$

Hence

$$y + c = \int \frac{p^2}{\sqrt{(1+p^2)}}dp$$

$$y + c = \frac{1}{2}\left(p\sqrt{1 + p^2} - \sinh^{-1}p\right) \text{ ---------------- (2.13)}$$

Eliminating p between (2.12) and (2.13), the solution is got.

Example 2: Solve $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$.

Solution:

Given the equation is $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$.

This is homogeneous in x and y and solvable for p.

$p = \frac{2y}{x}$ or -$\frac{3x}{y}$.

$\frac{dy}{y} = \frac{2dx}{x}$ or $ydy + 3xdx = 0$

$\therefore y = cx^2$ or $y^2 3x^2 = c$.

The solution is $(y - cx^2)(y^2 + 3x^2 - c) = 0$.

UNIT- III EQUATIONS HOMOGENEOUS IN x AND y

Structure:

3.1 EQUATION HOMOGENEOUS

Consider $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$, where f_1 and f_2 are homogeneous functions of the same degree in x and y.

$f_1(x,y)$ can be written as $x^n\phi(\frac{y}{x})$ and $f_2(x,y)$ as $x^n\psi(\frac{y}{x})$.

If we put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

The given equation becomes $v + x\frac{dv}{dx} = \frac{\phi(v)}{\psi(v)}$.

The variables can be separated; the equation is

$$\frac{dx}{x} + \frac{\psi(v)dv}{v\psi(v)-\phi(v)} = 0.$$

Integrating, $\log x + \int \frac{\psi(v)dv}{v\psi(v)-\phi(v)} = c$.

The solution is got by substituting $\frac{y}{x}$ for v after the integration is over.

Examples:

Example 1. Solve $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

Solution:

Given the equation is $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

Put $y = vx$; $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

The equation reduces to $\frac{dv(1-v)}{v} + \frac{dx}{x} = 0$.

Integrating, $\log v - v + \log x = \log c$.

The solution is $y = ce^{y/x}$.

Example 2. Solve $xdy - ydx = \sqrt{x^2+y^2}dx$.

Solution:

Given the equation is $xdy - ydx = \sqrt{x^2+y^2}dx$.

Put $y = vx$; $dy = vdx + xdv$.

The equation reduces to $\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$.

Integrating, $\log(v + \sqrt{1+v^2}) = \log x + \log c$.

The solution is $y + \sqrt{x^2+y^2} = cx^2$.

Exercises:

Solve the following equations:-

1. $\frac{dy}{dx} = \frac{x-y}{x+y}$.
2. $(y^2 - 2xy)dx = (x^2 - 2xy)dy$.
3. $(x+y)^2dx = 2x^2dy$.

3.2 LINEAR EQUATION WITH CONSTANT COEFFICIENTS

Introduction:

A linear equation of n^{th} order with constant coefficients is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a^n y = X \text{ --------------------- (3.1)}$$

where $a_1, a_2, a_3, \ldots . a_n$ are constants and X is a function of x. This equation can also be written in the form

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots . + a_n y = X$$

where $D = \frac{d}{dx}$; $D^2 = \frac{d^2}{dx^2}$ .... $D^n = \frac{d^n}{dx^n}$.

Result 1:

Consider $(D^n + a_1 d^{n-1} + a_2 D^{n-2} + \ldots . + a_n)y = 0$. -------------- (3.2)

If $y = y_1(x)$, $y = y_2(x)$ .... $y = y_n(x)$ are solutions of (3.2) then $c_1 y_1 + c_2 y_2 + \ldots . + c_n y_n$ is also a solution of (3.2) where $c_1, c_2, \ldots .., c_n$ are arbitrary constants.

Result 2:

If $c_1 y_1 + c_2 y_2 + \ldots . + c_n y_n$ is the general solution of (3.2) and if $y = u$ is a particular solution of (3.1) then $c_1 y_+ c_2 y_2 + \ldots . + c_n y_n + u$ be the general solution of (3.1).

Definition:

The general solution of (1) is of the form $y = Y + u$ where $Y = c_1 y_1 + c_2 y_2 + \ldots .. + c_n y_n$ is the general solution of (3.2). Y is called the complementary function (C.F) and u is called a particular integral (P.I).

Methods of finding complementary functions:

Consider the differential equation $y'' + ay' + by = 0$ --------(3.3) where a, b are constants.

case i: Roots of the auxiliary equation are real and distinct say m_1 and m_2. Then the general solution of (3.3) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

case ii: Roots of the auxiliary equation are imaginary. Then the solution of (1) is $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$.

case iii: Roots of the auxiliary equation are real and equal say $m_1 = m_2$.Then the general solution is $y = e^{m_1 x}(c_1 x + c_2)$.

3.3 PROBLEMS

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Problem 1. Solve $(D^2 - 5D + 6)y = 0$

Solution:

The auxiliary equation is $m^2 - 5m + 6 = 0$.
(ie.,) $(m-3)(m-2) = 0$.
Hence $m = 2,3$
The complementary function is C.F= $c_1e^{3x} + c_2e^{2x}$.
The general solution is given by $y = c_1e^{3x} + c_2e^{2x}$.

Problem 2. Solve $(D^2 + D + 1)^2y = 0$.

Solution:

The auxiliary equation is $(m^2 + m + 1)^2 = 0$.
(ie.,) $(m^2 + m + 1)(m^2 + m + 1) = 0$.
Hence $m = \frac{-1 \pm i\sqrt{3}}{2}$ (twice)
The complementary function is
C.F = $e^{-x/2}[(c_1 + c_2x)\cos(\sqrt{3}/2)x + (c_3 + c_4x)\sin(\sqrt{3}/2)x]$.
The general solution is given by
$e^{-x/2}[(c_1 + c_2x)\cos(\sqrt{3}/2)x + (c_3 + c_4x)\sin(\sqrt{3}/2)x]$.

UNIT - IV LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

Structure:

4.1 Linear Equations With Variable Coefficients.

4.2 Equations Reducible to the Linear Equations.

4.1 Linear equations with variable coefficients

A homogeneous linear equation is of the form

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \ldots + p_n y = X,$$

where $p_1, p_2, \ldots, p_n$ are constants and X a function of x.

Examples

Example 1. Solve $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x + \log x$.

Solution:

Given the equation is $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x + \log x$.

Putting $z = \log x$ and $D = \frac{d}{dz}$ the equation becomes

$(D^3 + 1)y = e^z + z$.

The auxiliary equation is $m^3 + 1 = 0$. $\therefore m = -1$ or $\frac{1 \pm \sqrt{3} i}{2}$.

C.F = $C_1 e^{-z} + e^{z/2}(C_2 \cos \frac{\sqrt{3}}{2} z + C_3 \sin \frac{\sqrt{3}}{2} z)$.

C.F = $C_1 x^{-1} + \sqrt{x}[C_2 \cos(\frac{\sqrt{3}}{2} \log x) + C_3 \sin(\frac{\sqrt{3}}{2} \log x)]$.

P.I. $= \frac{1}{D^3+1}(e^z + z)$

$= \frac{1}{2} e^z + (1 - D^3) z$

$= \frac{x}{2} + \log x$.

y=C.F+P.I.

Example 2. Solve $x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{(1-x)^2}$.

Solution:

Given the equation is $x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{(1-x)^2}$.

Putting $z = \log x$ and $D = \frac{d}{dz}$ the equation becomes

$(D^2 + 2D + 1)y = \frac{1}{(1-x)^2}$.

The auxiliary equation is $(m+1)^2 = 0$. $\therefore m = -1$ (twice)

C.F. = $e^- z(A + Bz) = \frac{1}{x}(A + B \log x). P.I. = \frac{1}{(\theta+1)^2} \frac{1}{(1-x)^2}$ changing D to the operator $\theta = x \frac{d}{dx}$

$\frac{1}{\theta+1} \frac{1}{x} \frac{1}{1-x}$

$= \frac{1}{x} \log \frac{x}{1-x}$. y=C.F+P.I.

4.2 EQUATIONS REDUCIBLE TO THE LINEAR EQUATIONS

Consider an equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + (a+bx)^{n-1} p_1 \frac{d^{n-1}}{dx^{n-1}} + \ldots + p_n y = X,$$

where $p_1, p_2, \ldots., p_n$ are constants and X is any function of x.

Putting $z = a + bx$, the equation transforms into

$$z^n \frac{d^n y}{dz^n} + \frac{p_1 z^{n-1}}{b} \frac{d^{n-1} y}{dz^{n-1}} + \ldots. + \frac{p_n}{b^n} y = \frac{1}{b^n} X(\frac{z-a}{b}.$$

This is linear homogeneous equation.

Example: Solve $(5+2x)^2 \frac{d^2 y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 6x.$

Solution:

Given the equation is $(5+2x)^2 \frac{d^2 y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 6x.$

Putting $z = 5 + 2x$, the equation becomes

$$4z^2 \frac{d^2 y}{dz^2} - 12z\frac{dy}{dz} + 8y = 3(z-5)$$

.

Putting $u = \log z$ and $D = \frac{d}{du}$,

the equation is now transformed into

$(4D^2 - 16D + 8)y = 3(e^u - 5).$

The auxiliary equation is $m^2 - 4m + 2 = 0.$

$\therefore m = 2 \pm \sqrt{2}.$

C.F= $(5+2x)^2[A(5+2x)^{\sqrt{2}} + B(5+2x)^{\sqrt{2}}].$

P.I= $\frac{3}{4(D^2-4D+2)}(e^u - 5)$

P.I= $\frac{-3}{4}z - \frac{15}{8}$

P.I= $\frac{-3}{2}x - \frac{45}{8}$

y=C.F+P.I.

UNIT-V SIMULTANEOUS DIFFERENTIAL EQUATIONS

Structure

5.1 Introduction

5.2 Simultaneous equations of the first order and first degree

5.2.1 Methods for solving $\frac{\mathbf{d}\boldsymbol{x}}{\boldsymbol{P}}=\frac{\mathbf{d}\boldsymbol{y}}{\boldsymbol{Q}}=\frac{\mathbf{d}\boldsymbol{z}}{\boldsymbol{R}}$

5.3 Simultaneous linear differential equations

5.1 INTRODUCTION

In this chapter, we shall discuss differential equations is which there is one independent variable and two or more than two dependent variables. To solve such equations completely, there must be as many equations as there are dependent variables. Such equations are called its *ordinary simultaneous differential equations*

5.2 Simultaneous equations of the first order and first degree

Taking z as the independent variable and x and y as the pair of dependent variables, a pair of simultaneous differential equations of the first order and first degreemay be written as

$$P_1\frac{\mathrm{d}x}{\mathrm{d}z}+Q_1\frac{\mathrm{d}y}{\mathrm{d}z}+R_1=0$$
$$P_2\frac{\mathrm{d}x}{\mathrm{d}z}+Q_2\frac{\mathrm{d}y}{\mathrm{d}z}+R_2=0$$

where $P_1,\ Q_1,\ R_1,\ P_2,\ Q_2$ and R_2 are the functions of $x,\ y$ and z. These equations can be written as

$$P_1\mathrm{d}x+Q_1\mathrm{d}y+R_1\mathrm{d}z=0$$
$$P_2\mathrm{d}x+Q_2\mathrm{d}y+R_2\mathrm{d}z=0$$

The ratios of the differentials $\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z$ can be obtained as

$$\frac{\mathrm{d}x}{Q_1R_2-Q_2R_1}=\frac{\mathrm{d}y}{R_1P_2-R_1P_1}=\frac{\mathrm{d}z}{P_1Q_2-P_2Q_1}$$

$$or\quad \frac{\mathrm{d}x}{P}=\frac{\mathrm{d}y}{Q}=\frac{\mathrm{d}z}{R}$$

where $P,\ Q,\ R$ are the functions of $x,\ y$ and z.

We shall take (5.3) as the standard form for a pair of ordinary simultaneous equations of the first order and first degree.

Two independent relations between the variables $x,\ y$ and z each involving an arbitrary constant constitute the general solution of equations (5.3) and so of equations (5.3) if and only if $\frac{\mathrm{d}x}{\mathrm{d}z}$ and $\frac{\mathrm{d}y}{\mathrm{d}z}$ deducible from them satisfy equations (5.3) and so (5.3).

5.2.1 Methods for solving $\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}$

Case i. Whentwo of the ratios in equation (5.3) involve only the two corresponding out of the three variables x, y and z.

Suppose that the equation $\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q}$ does not involve z or that it can be reduced to a differential equation in x and y only by the cancellation of a common factor in P and Q.

In that case the equation $\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q}$ is of the first order and first degree and the equation may be solved. Let its solution be $u = c_1$. It will be a part of the general solution of equations (5.3) since

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0 \ \textit{and therefore} \tag{5.4}$$

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0 \tag{5.5}$$

We might, either by making use of the relation $u = c_1$ or otherwise independently, be similarly able too obtain another relation $v = c_2$ which is also a part of the general solution of equations (5.3). These two relations, if they are independent of each other, will constitute the general solution of equations (5.3).

Case ii. A part or the whole of the general solution of equations (5.3) can also be found by the following method:-

Each of the ratios in equations (5.3) is equal to

$$\frac{l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z}{lP + mQ + nR}$$

where l, m and n are any multipliers; and hence, if the multipliers l, m and n are such that $lP + mQ + nR = 0$, we must then also have $l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z = 0$.

If the left-hand side of this equation can be expressed as the differential $\mathrm{d}u$ of some function u of x, y and z then $u = c_1$ is a solution of the equation $l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z = 0$ if l, m and n are proportional to

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial z}.$$

Since $lP + mQ + nR = 0$, $\quad P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0$.

Hence $u = c_1$ is a part of the general solution of equation (5.3).

A very simple case in which u can be obtained immediately from equation $l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z = 0$ that where l is a function of x alone, m of y alone and n of z alone. In that case

$$u = \int (l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z)$$

If two essentially different sets of such multipliers l, m and n can be obtained, then the general solution of equation (5.3) can be deduced by this method.

Examples:

1. Solve $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$

   Taking the first two equations $\frac{dx}{x}=\frac{dy}{y}$, we get $x=c_1y$.

   Taking the first and last equations $\frac{dx}{x}=\frac{dz}{z}$, we get $x=c_2z$.

   Thus the required solution is given by $x=c_1y$ and $x=c_2z$.

2. Solve the equations $\frac{dx}{yz}=\frac{dy}{xz}=\frac{dz}{xy}$.

Solution:

Taking the first two equations, we get $\frac{dx}{yz}=\frac{dy}{xz}$, *i.e.*, $\frac{dx}{y}=\frac{dy}{x}$

i.e., $x\mathrm{d}x-y\mathrm{d}y=0$ *i.e.*, $x^2-y^2=c_1$

Taking the first and the last equations, we get

$\frac{dx}{yz}=\frac{dz}{xy}$, *i.e.*, $\frac{dx}{z}=\frac{dz}{x}$

i.e., $x\mathrm{d}x-z\mathrm{d}z=0$ *i.e.*, $x^2-z^2=c_2$

$\therefore$ The general solution is $\phi(x^2-y^2, x^2-z^2)=0$.

2. Solve the equations $\frac{dx}{-y^2-z^2}=\frac{dy}{xy}=\frac{dz}{xz}$.

Solution:

The equation $\frac{dy}{xy}=\frac{dz}{xz}$ gives $\frac{dy}{y}=\frac{dz}{z}$

i.e., $\log y=\log z+\log c_1$, *i.e.*, $y=c_1z$

Considering each of the ratios in the given equation

$\frac{dx}{-y^2-z^2}=\frac{dy}{xy}=\frac{dz}{xz}$, we get $\frac{x\mathrm{d}x+y\mathrm{d}y+z\mathrm{d}z}{x(-y^2-z^2)+xy^2+xz^2}=\frac{x\mathrm{d}x+y\mathrm{d}y+z\mathrm{d}z}{0}$

$\therefore\ x\mathrm{d}x+y\mathrm{d}y+z\mathrm{d}z=0,\ \therefore\ x^2+y^2+z^2=c_2$.

Hence the general solution is $\phi\left(\frac{y}{z}, x^2+y^2+z^2\right)=0$

Exercises

Solve the following equations:

1. $\frac{dx}{x^2+y^2}=\frac{dy}{2xy}=\frac{dz}{(x+y)z}$
2. $\frac{dx}{y^2-z^2+x^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$.
3. $\frac{dx}{y+z}=\frac{dy}{x+z}=\frac{dz}{x+y}$.
4. $\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$.
5. $\frac{dx}{y+zx}=\frac{dy}{-x-yz}=\frac{dz}{x^2-y^2}$.

5.3 Simultaneous linear differential equations

We now treat systems of equations. There is only one independent variable usually denoted by the symbol t and a number of dependent variables equal to the number of equations. We shall further suppose that the equations are linear.

Let D stands for $\frac{d}{dt}$. Taking the simplest case of two dependent variables x and y, the equations can be written in the form

$$f_1(D)x + \phi_1(D)y = T_1 \quad (5.4)$$

$$f_2(D)x + \phi_2(D)y = T_2 \quad (5.5)$$

(5.4)
(5.5)

where we shall suppose that f_1, f_2, ϕ_1 and ϕ_2 are rational integral functions of D with constant coefficients and T_1, T_2 explicit functions of t. To eliminate y as in solving simultaneous algebraic equations we operate on (5.4) by $\phi_2(D)$ and (5.5) by $\phi_1(D)$ and subtract. Then we have

$$\{f_1(D)\phi_2(D) - f_2(D)\phi_1(D)\}x = \phi_2(D)T_1 - \phi_1(D)T_2$$

Substituting for x in (5.4) or (5.5), y can be found.

It must be noted that the number of arbitrary constants in the complete solution is the exponent of the highest index in the operator D of $f_1(D)\phi_2(D) - f_2(D)\phi_1(D)$.

The above method can be extended to equations of more than two dependent variables. The following examples illustrate the process clearly.

Examples:

1. Solve the system

$$\frac{dx}{dt} + 2x - 3y = t \quad (5.6)$$

(5.6)

and

$$\frac{dy}{dt} - 3x + 2y = e^{2t} \quad (5.7)$$

Solution

The equations can be written as

$$(D+2)x - 3y = t \quad and \quad (D+2)y - 3x = e^{2t}$$

To eliminate y, operate on (5.6) by $D+2$ and (5.7) by 3 and add. We get

$$((D+2)^2 - 9)x = (D+2)t + 3e^{2t}$$

$$i.e., (D^2 + 4D - 5)x = 1 + 2t + 3e^{2t}$$

The auxillary equation is $m^2 + 4m - 5 = 0$.

$\therefore m = 1$ or -5.

C.F. is $C_1e^t + C_2e^{-5t}$.

$$P.I. = \frac{1}{D^2+4D-5}(1 + 2t + 3e^{2t})$$

$$= \frac{1}{5}\left(1 + \frac{4D}{5}\right)(1 + 2t) + \frac{3e^{2t}}{7}$$

$$= \frac{13}{25} - \frac{2t}{5} + \frac{3e^{2t}}{7}.$$

$\therefore\ x = C_1e^t + C_2e^{-5t} + \frac{13}{25} - \frac{2t}{5} + \frac{3e^{2t}}{7}$

Thus $y = C_1e^t - C_2e^{-5t} - \frac{12}{25} - \frac{3t}{5} + \frac{4e^{2t}}{7}$.

2. Solve the system

$$4\frac{dx}{dt} + 9\frac{dy}{dt} + 2x + 31y = e^t$$

and

$$3\frac{dx}{dt} + 7\frac{dy}{dt} + x + 24y = 3$$

Solution

The equations can be written as

$2(2D+1)x + (9D+31)y = e^t \quad and \quad (3D+1)x + (7D+24)y = 3.$

Eliminating y, we get $(D^2+8D+17)x = 31(e^t - 3)$.

The auxillary equation is $m^2 + 8m + 17 = 0$.

$\therefore\quad m = -4 \pm i$.

$\therefore\quad C.F. = e^{-4t}(C_1\sin t + C_2\cos t)$.

$P.I. = \frac{1}{D^2+8D+17}31(e^t - 3) = \frac{31}{26}e^t - \frac{93}{17}$.

$\therefore\quad x = e^{-4t}(C_1\sin t + C_2\cos t) + \frac{31}{26}e^t - \frac{93}{17}$.

Similarly, eliminating x, we have

$(D^2 + 8D + 17)x = 5 - 4e^t$.

$\therefore\quad y = e^{-4t}(C_3\cos t + C_4\sin t) + \frac{6}{17} - \frac{2e^t}{13}$.

The relations between the constants are got by substituting these values in the two equations. Substituting, we get

$\sin t(-14C_1 - 4C_2 - 5C_4 + 9C_3) + \cos t(-14C_3 + 4C_1 + 9C_4 - 5C_3) = 0,$

whence $C_3 = -(C_1 + C_2)$ and $C_4 = C_2 - C_1$.

$\therefore\quad y = [(C_2 - C_1)\sin t - (C_2 + C_1)\cos t]e^{-4t} - \frac{2e^t}{13} + \frac{6}{17}$.

3. Solve $\frac{dx}{dt} + 4x + 3y = t$ (

and

$\frac{dy}{dt} + 2x + 5y = e^t$ (

Solution

The equations can be written as

$(D+4)x + 3y = t \quad and \quad (D+5)y + 2x = e^t$

To eliminate y, operate on (5.10) by $D + 5$ and (5.11) by 3 and subtract. We get

$$(D^2 + 9D + 14)x = 1 + 5t - 3e^t$$

This is a linear differential equation with constant coefficients with the dependent variable x and independent variable t.

The auxillary equation is $m^2 + 9m + 14 = 0$.

$\therefore\ m = -7$ or -2.

C.F. is $C_1e^{-7t} + C_2e^{-2t}$.

$$P.I. = \frac{1}{D^2+4D+14}(1 + 5t - 3e^t)$$

$$= -\frac{211}{196} + \frac{25t}{14} - \frac{e^t}{8}.$$

$$\therefore\ x = C_1e^{-7t} + C_2e^{-2t} - \frac{211}{196} + \frac{25t}{14} - \frac{e^t}{8}.$$

Thus $y = \frac{1}{3}\left[3C_1e^{-7t} - 2C_2e^{-2t} + \frac{5e^t}{8} - \frac{48}{7}\mathrm{t} + \frac{247}{98}.\right]$. (5.11)

Exercises

Solve the following equations:

1. $\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by + c;\ \frac{\mathrm{d}y}{\mathrm{d}t} = a'x + b'y + c'$.
2. $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} - x + \sin t = 0;\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} - y + \cos t = 0$.
3. $3(1 - D)x + 4y = 3t + 1;\ 3(D + 1)y + 2x = e^t$
4. $(D + 5)x + y = e^t;\ (D + 3)y - x = e^{2t}$.
5. $(D + 8)x + (2D + 1)y = 0;\ (6D - 2)x - (3D - 11)y = 0$.

UNIT-VI LINEAR EQUATIONS OF THE SECOND ORDER

Structure

6.1 Complete solution given a known integral

Type I : If an integral included in the complementary function of the given equation be known, the complete solution can be found in terms of this known integral.

Let

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad (6.1)$$

be the given equation, where P, Q and R are the functions of x.

Let $y = y_1$ be a known integral in the C.F. of (6.1).

$$i.e., \frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

$$\therefore \quad \frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1 = 0 \qquad (6.2)$$

Putting $y = y_1 v$ in (6.1), where v is a function of x, we get

$$y_1\frac{d^2v}{dx^2} + \frac{dv}{dx}\left(2\frac{dy_1}{dx} + Py_1\right) = R$$

in virtue of (6.2). This is linear in $\frac{dv}{dx}$; Hence

$$\frac{dv}{dx} = \frac{c_1}{y_1^2}e^{-\int P dx} + \frac{e^{-\int P dx}}{y_1^2}\int Ry_1 e^{\int P dx} dx$$

Integrating,

$$v = c_2 + c_1\int \frac{e^{-\int P dx}}{y_1^2}dx + \int\left(\frac{e^{-\int P dx}}{y_1^2}\int Ry_1 e^{\int P dx} dx\right)dx.$$

The solution of (6.1) is $y = vy_1$, where v has the above value.

It must be noted that this solution includes the given solution and that there are two arbitrary constants.

Examples:

1. Solve $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$.

Solution

As the sum of the coefficients of the first member is zero, e^x is a solution of

$$x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x.$$

Putting $y = ve^x$, the given equation reduces to $x\frac{d^2v}{dx^2} + \frac{dv}{dx} = 1$.

Solving, $\frac{dv}{dx}x = x + c_1$. i.e.,$\frac{dv}{dx} = 1 + \frac{c_1}{x}$.

Integrating $v = x + c_1\log x + c_2$.

Hence $y = e^x(x + c_1 \log x + c_2)$.

2. Solve $x^2 \frac{d^2y}{dx^2} - (x^2 + 2x)\frac{dy}{dx} + (x + 2)y = x^3 e^x$.

Solution

$y = x$ is a solution of this equation without the second member,
Putting $y = vx$, the given equation reduces to $v_2 - v_1 = e^x$.
Hence, $v_1 e^x = x + c_1$ or $v_1 = (x + c_1)e^x$.
Integrating $v = (c_1 - 1)e^x + xe^x + c_2$.
$\therefore\ y = c_2 x + (c_1 - 1)xe^x + x^2 e^x$.

Type II:

Consider a differential equation of type $y'' + py' + qy = 0$,where p, q are some constant coefficients. For each of the equation we can write the so-called **characteristic (auxiliary) equation:** $k^2 + pk + q = 0$.

The general solution of the homogeneous differential equation depends on the roots of the characteristic quadratic equation. There are the following options:

1. Discriminant of the characteristic quadratic equation $D > 0$. Then the roots of the characteristic equations k_1 and k_2 are real and distinct. In this case the general solution is given by the following function
$$y(x) = C_1 e^{k_1 x} + C_2 e^{k_2 x}$$
where C_1 and C_2 are arbitrary real numbers.
2. Discriminant of the characteristic quadratic equation D=0. Then the roots are real and equal. It is said in this case that there exists one repeated root k_1 of order 2. The general solution of the differential equation has the form:
$$y(x) = (C_1 + C_2)e^{kx}$$
3. Discriminant of the characteristic quadratic equation $D < 0$. Such an equation has complex roots $k_1 = \alpha + i\beta,\ k_2 = \alpha - i\beta$. The general solution is written as $y(x) = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x]$.

Examples:

1. Solve the differential equation $y''-6y'+5y=0$.

Solution

First we write the corresponding characteristic equation for the given differential equation:

$k^2 - 6k + 5 = 0.$

The roots of this equation are $k = 1, 5$. Since the roots are real and distinct, the general solution has the form:

$$y(x) = C_1 e^x + C_2 e^{5x}$$

where C_1 and C_2 are arbitrary constants.

2. Find the general solution of the equation $y'' - 6y' + 9y = 0$.

Solution

We write the characteristic equation and calculate its roots:

$k^2 - 6k + 9 = 0$.As it can be seen, the characteristic equation has one root of order 2: $k = 3$. Therefore, the general solution of the differential equation is given by

$$y(x) = (C_1 + C_2)e^{3x}$$

where C_1 and C_2 are arbitrary real numbers.

3. Solve the equation $y'' + 25y = 0$.

Solution

The characteristic equation has the form:

$k^2 + 25 = 0.$

This equation has pure imaginary roots:

$k = \pm 5i.$

Then the answer can be written as follows:

$y(x) = C_1 \cos 5x + C_2 \sin 5x,$

where C_1, C_2 are constants of integration.

4. Solve the equation $y'' + 4iy = 0$.

Solution

In this equation the coefficient before y is a complex number. The general solution for linear differential equations with constant complex coefficients is constructed in the same way. First we write the characteristic equation:

$k^2 + 4i = 0.$

Determine the roots of the equation:

$k^2 = -4i, \implies k = \pm 2i\sqrt{i}$.

Calculate separately the square root of the imaginary unit. It is convenient to represent the number i in trigonometric form:

$$\sqrt{i} = \sqrt{(\cos 2\pi + i \sin 2\pi)} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

Then the roots of the characteristic equation are given by
$k_1 = -\sqrt{2} + i\sqrt{2}$, $k_2 = \sqrt{2} - i\sqrt{2}$.
The general solution of the initial differential equation will be expressed through the linear combination of the exponential functions with the found complex numbers:

$$y(x) = C_1 e^{(-\sqrt{2}+i\sqrt{2})x} + C_2 e^{(\sqrt{2}-i\sqrt{2})x}$$

where C_1, C_2 are arbitrary constants.

Exercises

Solve the following equations:

1. $x\frac{d^2y}{dx^2} + 2(4x-1)\frac{dy}{dx} - (9x-2)y = x^3e^x.$
2. $xy_2 + (1-x)y_1 - y = e^x.$
3. $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$
4. $(x\sin x + \cos x)\frac{d^2y}{dx^2} - x\cos x\frac{dy}{dx} + y\cos x = 0$
5. $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + x^2y = 0.$
6. $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + xy = 0.$
7. $y'' + y' - 6y = 0.$
8. $3\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0.$
9. $y'' - 6y' + 13y = 0.$
10. $y'' + 2y' + y = 0.$

UNIT VII REDUCTION TO NORMAL FORM

Structure

7.1 Reduction to normal form

Consider

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad (7.1)$$

Putting $y = y_1 v$, this becomes

$$y_1.\frac{d^2v}{dx^2} + \frac{dv}{dx}(2\frac{dy_1}{dx} + py_1) + v\{\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1\} = R$$

If y_1 be chosen to satisfy

$$2\frac{dy_1}{dx} + py_1 = 0$$

i.e., $y_1 = e^{-\frac{1}{2}\int Pdx}$, then the above equation becomes,

$$\frac{d^2v}{dx^2} + Iv = Re^{\frac{1}{2}\int Pdx} \quad (7.2)$$

Where $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$.

Equation 7.2 is immediately integrable if I be either a constant or $\frac{a}{x^2}$, where a is a constant.

This method is either called reducible equation 7.1 to the normal form or removing the first derivative.

Examples:

Example 1:

Solve $y_2 - 4xy_1 + (4x^2 - 3)y = e^{x^2}$.

Solution:

Here $P = -4x,\ Q = 4x^2 - 3$ and $R = e^{x^2}$

$$y_1 = e^{-\frac{1}{2}\int Pdx} = e^{x^2}$$

putting $y = vy_1$ $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = -1$ and the equation reduces to $\frac{d^2v}{dx} - v = 1$.

Hence $v = Ae^x + Be^{-x} - 1$

Therefore $y = e^{x^2}(Ae^x + Be^{-x} - 1)$.

Example 2:

Solve $4x^2\frac{d^2y}{dx^2} + 4x^5\frac{dy}{dx} + (x^8 + 6x^4 + 4)y = 0$.

Here $P = x^3;\ Q = \frac{1}{4}(x^6 + 6x^2 + \frac{4}{x^3})$.

Hence

$$y_1 = e^{-\frac{1}{2}\int Pdx} = e^{-\frac{x^4}{8}}$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = \frac{1}{x^2}.$$

Hence the equation in v, where $y = vy_1$, is

$$\frac{d^2v}{dx^2}+\frac{1}{x^2}v=0$$

This is a homogeneous linear equation whose solution is

$$\sqrt{x}A\cos\left(\frac{\sqrt{3}}{2}\log x+B\right)$$

Example 3:

Solve $\left(\frac{d^2y}{dx^2}\right)-\left(\frac{dy}{dx}\right)2\tan x+5y=0$

Solution:

Here $P=-2\tan x$; $Q=5$ and $R=0$.

To remove the first derivative we choose $u=e^{-\frac{1}{2}\int Pdx}=e^{-\int(-2\tan x dx)dx}=e^{\log\sec x}=\sec x$.

Put $y=vy_1=v\sec x$.

$$I=Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}=5+\sec^2x-\tan^2x=6$$

$\therefore$ The given equation reduces to $\frac{d^2v}{dx^2}+6v=0$ which is a linear equation with constant coefficients.

The auxillary equation is $m^2+6=0$. Hence $m=\pm i\sqrt{6}$.

$\therefore y_1=c_1\cos\sqrt{6}x+c_2\sin\sqrt{6}x$.

$\therefore$ The general solution of the given equation is

$$y=vy_1=\sec x(c_1\cos\sqrt{6}x+c_2\sin\sqrt{6}x)$$

Example 4:

Solve $\frac{d^2y}{dx^2}+\frac{1}{x^{\frac{1}{3}}}\frac{dy}{dx}+\left(\frac{1}{4x^{\frac{2}{3}}}-\frac{1}{6x^{\frac{4}{3}}}-\frac{6}{x^2}\right)y=0$ by removing the first derivative.

Here $P=x^{-1/3}$; $Q=\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}$ and $X=0$.

To remove thr first derivative we choose

$u=e^{-\frac{1}{2}\int Pdx}=e^{-\frac{1}{2}\int x^{-\frac{1}{3}}dx}=e^{-\frac{3}{4}x^{\frac{2}{3}}}$

Put $y=uv$

The given equation reduces to $v''+Q_1v=0$

where $Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}P'=-\frac{6}{x^2}$

$\therefore \frac{d^2v}{dx^2}-\frac{6v}{x^2}=0$.

$\therefore x^2\frac{d^2v}{dx^2}-6v=0$ which is a homogenous linear equation.

Put $x=e^z$. The equation is transformed to $[\theta(\theta-1)-6]v=0$ where $\theta=\frac{d}{dz}$.

$\therefore (\theta^2-\theta-6)v=0$.

The A.E is $m^2-m-6=0$. Hence $m=-2$ or 3.

$\therefore v=c_1x^3+\frac{c_2}{x^2}$

Hence the solution of the given equation is

$y=uv=e^{-\frac{3}{4}x^{\frac{2}{3}}}\left(c_1x^3+\frac{c_2}{x^2}\right)$.

Exercise

Solve the following by removing the first derivative:

(i) $(\frac{d^2y}{dx^2}) + 4x(\frac{dy}{dx}) + 4x^2y = 0$

(ii) $(\frac{d^2y}{dx^2}) - \frac{2}{x}(\frac{dy}{dx}) + (1 + \frac{2}{x^2}y) = xe^x$

(iii) $(\frac{d^2y}{dx^2}) + 4x(\frac{dy}{dx}) + 4(x^2 - 1)y = -3e^{x^2}sin2x$

(iv) $(\frac{d^2y}{dx^2}) + (2/x)(\frac{dy}{dx}) + n^2y = 0$

7.2 Change of the independent variable

Let z be the new independent variable.

Consider

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad (7.3)$$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}; \frac{d^y}{dx^2} = \frac{d^2y}{dz^2}(\frac{dz}{dx})^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

The equation 1.3 transforms into

$$\frac{d^2y}{dz^2}(\frac{dz}{dx})^2 + \frac{dy}{dz}(\frac{d^2z}{dx^2} + P\frac{dz}{dx}) + Qy = R \qquad (7.4)$$

We may choose z such that the coefficient of $\frac{dy}{dz}$,

$$i.e., \frac{d^2z}{dx^2} + P\frac{dz}{dx}\ vanishes \qquad (7.5)$$

Therefore $z = \int e^{-\int Pdx}dx$

Using this value of z, equation 7.4, may become integrable. One particular case where equation 7.4 becomes immediately integrable is when

$Q = \mu(\frac{dz}{dx})^2$, where μ is constant. Then equation 7.4 reduces to

$$\frac{d^y}{dz^2} + \mu y = \frac{R}{(\frac{dz}{dx})^2}$$

The second case, when equation 16 becomes integrable, is when

$$Qz^2 = \mu(\frac{dz}{dx})^2$$

Then equation 7.4 becomes a homogeneous linear equation.

Example 1:

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx}tanx + ycos^2x = 0.$

Solution:

Here $P = tanx$ and $Q = cos^2x.$

Choosing $z = \int e^{\int -tanxdx}dx = sinx$

The equation transforms to $\frac{d^2y}{dz^2} + y = 0$ as here $Q = cos^2x = (\frac{dz}{dx})^2.$

$$Therefore\ y = Acosz + Bsinz. = Acos(sinx) + Bsin(sinx).$$

Example 2:

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx}cotx + 4ycosec^2x = 0$ by changing the independent variable x to $z.$

Solution:

Here $P = cotx;\ Q = 4cosec^2x;\ R = 0.$

To change the independent variable x to z choose z such that

$$z = \int e^{-\int Pdx}dx$$
$$\therefore z = e^{-\int cotxdx}dx$$
$$= \int e^{logcosecx}dx$$
$$= \int cosecxdx$$
$$= logtan(x/2).$$

With this choice of z the given equation reduces to

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \ where\ P_1 = (\frac{dz}{dx})^{-2}(\frac{d^2z}{dx^2} + P\frac{dz}{dx}) = 0$$

$$Q_1 = Q(\frac{dz}{dx})^{-2} = 4; R_1 = R(\frac{dz}{dx})^{-2} = 0.$$

Hence the equation is transformed to $\frac{d^2y}{dz^2} + 4y = 0$ which is a linear second order differential equation with constant coefficients, whose solution is $t = c_1cos2z + c_2sin2z.$

$\therefore$ The general solution is

$$y = c_1cos(2logtan(x/2)) + c_2sin(2logtan(x/2)).$$

Example 3:

Solve $y'' + y'cotx + 4ycosec^2x = 0$ by changing the independent variable x to z.

Solution:

Here $P = \cot x;\ \ Q = 4\,cosec^2x;\ \ \ X = 0.$

To change the independent variable x to z such that

$$z = \int e^{-\int Pdx}dx$$

$$z = \int e^{-\int cotxdx}dx$$
$$z = \int e^{\log cosecx}dx$$
$$= \int cosecx\,dx = \log\tan\left(\frac{x}{2}\right).$$

With this choice of z the given equation reduces to

$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = X_1$ where $P_1 = \left(\frac{dz}{dx}\right)^{-2}\left[\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right] = 0.$

$$Q_1 = Q\left(\frac{dz}{dx}\right)^{-2} = 4;\quad X_1 = X\left(\frac{dz}{dx}\right)^{-2} = 0$$

Hence the equation is transformed to $\frac{d^2y}{dz^2} + 4y = 0$ which is a linear second order differential equation with constant coefficients, whose solution is $y = c_1\cos 2z + c_2\sin 2z.$

$\therefore$ The general solution is

$y = c_1\cos(2\log\tan(\frac{x}{2})) + c\_2\sin\ (2\log\tan(\frac{x}{2})).$

Example 4:

Solve $x^6y'' + 3x^5y' + a^2y = x^{-2}$.
Solution:
The given equation can be written in the form
Here $P = \left(\frac{3}{x}\right)$; $Q = \left(\frac{a^2}{x^6}\right)$; $X = \left(\frac{1}{x^8}\right)$
To change the independent variable x to z choose z such that $z = \int e^{-\int P dx} dx$.
$\therefore z = \int e^{-\int \left(\frac{3}{x}\right)dx} dx = \int e^{\log x^{-3}} dx = \int x^{-3} dx = -\frac{1}{2}x^{-2}$

$$z = -\frac{1}{2}x^{-2}; z' = x^{-3}; \; z'' = -3x^{-4}$$

With this choice of z the given equation is transformed to
$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = X_1$
where $P_1 = \left(\frac{dz}{dx}\right)^{-2} = a^2$ and
$X_1 = X\left(\frac{dz}{dx}\right)^{-2} = \frac{2}{z}$.
Hence the equation is transformed to $\frac{d^2y}{dz^2} + a^2y = -2z$ which is linear second order differential equation with constant coefficients whose C.F is $y = c_1 \cos az + c_2 \sin az$ and
$P.I = \frac{1}{D^2+a^2}(-2z)$ where $D = \frac{d}{dz}$

$$= \frac{1}{a^2}\left(1 + \frac{D^2}{a^2}\right)(-2z)$$

$$= -\frac{2z}{a^2}$$

Therefore, the general solution is given by

$$y = c_1 \cos\left(-\frac{a}{2x^2}\right) + c_2 \sin\left(-\frac{a}{2x^2}\right) + \left(\frac{1}{a^2x^2}\right).$$

Exercise

(i) $(\frac{d^2y}{dx^2}) + (2/x)(\frac{dy}{dx}) + (a^2/x^4)y = 0$

(ii) $(\frac{d^2y}{dx^2}) - cotx(\frac{dy}{dx}) + ysin^2x = 0$

(iii) $(\frac{d^2y}{dx^2}) - (1/x)(\frac{dy}{dx}) + 4x^2y = x^4$

(iv) $(\frac{d^2y}{dx^2}) + (\frac{dy}{dx})tanx - 2ycos^2x = cos^4x.$

UNIT VIII METHOD OF VARIATION OF PARAMETERS

Structure

8.1 Method of Variation of Parameters
8.2 Total differential equations
8.3 Methods of solving the differential equation
8.4 Solved problems

8.1 Method of Variation of Parameters

The method of variation of parameters is another method for solving a linear differential equation, either of the first order or of the second order. If the given equation is of the form $f(D)y = x$, this method can be applied to get the general solution, provided the corresponding homogeneous equation, viz., $f(D)y = 0$ can be solved by earlier methods. The procedures to solve linear equations of the first and second orders are the following.

$$\textit{Solution of the equation } \frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad (8.1)$$

where P, Q and R are functions of x or constants.

The homogeneous equation corresponding to equation 8.1 is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad (8.2)$$

Let the general solution of equation 8.2 to be

$$y = Au + Bv \qquad (8.3)$$

where A and B are arbitrary constants (parameters) and $y = u(x)$ and $y = v(x)$ are independent particular solutions of equation (8.2).

Now, we treat A and B as functions of x and assume equation 8.3 to be the general solution of equation 8.1. Differentiating equation 8.3 with respect to x, we have

$$\frac{dy}{dx} = (Au' + Bv') + (A'u + B'v) \qquad (8.4)$$

We choose A and B such that

$$A'u + B'v = 0 \qquad (8.5)$$

Then equation 8.4 becomes

$$\frac{dy}{dx} = Au' + Bv' \qquad (8.6)$$

Differentiating equation 8.6 with respect to x, we have

$$\frac{d^2y}{dx^2} = Au'' + Bv'' + A'u' + B'v' \qquad (8.7)$$

Since equation 8.3 is a solution of equation 8.1, 8.3, 8.6 and 8.1 satisfy equation 8.1.

$$(Au'' + Bv'' + A'u' + B'v') + P(Au' + Bv') + Q(Au + Bv) = R$$

$$A(u'' + Pu' + Qu) + B(v'' + Pv' + Qv) + A'u' + B'v' = R \qquad (8.8)$$

since $y = u$ is a solution of (8.2)

$$u'' + Pu' + Qu = 0$$

$$\textit{similarly } v'' + Pv' + Qv = 0$$

Inserting these values in equation 8.8, it reduces to

$$A'u' + B'v' = R \qquad (8.9)$$

Solving equation 8.5 and 8.9, we get the values of A' and B' integrating which, we get the values of A and B as functions of x. Using these values in equation 8.3, we get the required general solution of equation 8.1.

Working rule:

Let the given equation by $y'' + Py' + Qy = R$.

(i) Let the complementary function of (8.1) be $au + bv$.

(ii) Take the general solution as $y = Au + Bv$ where A and B are functions of x to be determined.

(iii) Form the equations $A'u + B'v = 0$ and $A'u' + B'v' = R$; Solve for A' and B'.

(iv) Get A and B on integration. Then the general solution is given by $y = Au + Bv$.

Examples

Example 1:

Solve the equation $(x^2+1)\frac{dy}{dx} + 4xy = \frac{1}{x^2+1}$, by using the method of variation of parameters.

The homogeneous equation corresponding to the given equation is

$$(x^2+1)\frac{dy}{dx} + 4xy = 0 \qquad (8.10)$$

$$\frac{dy}{y} + \frac{4x}{x^2+1}dx = 0$$

Integrating, we get $log\, y + 2\, log\,(x^2+1) = log\, c$

$$y = \frac{c}{(x^2+1)^2} \qquad (8.11)$$

$is\ the\ solution\ of\ equation$ 8.10

Treating c as a function of x and differentiating equation 8.11 with respect to x, we have

$$\frac{dy}{dx} = \frac{(x^2+1)^2 c' - c.2(x^2+1)2x}{(x^2+1)^4} \qquad (8.12)$$

Using equation 8.11 and 8.12 in the given equation, we have

$$\frac{(x^2+1)^2 c' - 4cx(x^2+1)}{(x^2+1)^2} + \frac{4cx}{(x^2+1)^2} = \frac{1}{x^2+1}$$

$$i.e. (x^2+1)^2 c' - 4cx(x^2+1) + 4cx(x^2+1) = (x^2+1)^2$$

$$i.e. c' = 1$$

$$\therefore c = x + k \qquad (8.13)$$

Using equation 8.13 in 8.11, the required solution of the given equation is $(x^2+1)^2 y = x + k$, where k is an arbitrary constant.

Example 2:

Using the method of variations of parameters solve $(D^2+1)y = x$.

Solution:

The auxiliary equation is $m^2 + 1 = 0$. Hence $m = \pm i$.

The complementary function = $a\cos x + b\,sinx$.

Let $u = cosx + bsinx$.

∴ The general solution is given by

$y = Au + Bv$

$\quad = A\,cosx + B\,sinx$

Where A and B are functions of x determined by the equations

$A'u + B'v = 0$ and $A'u' + B'v' = R$.

Now $A'u + B'v = 0$

$A'cosx + B'sinx = 0$

$A'u' + B'v' = R$

$-A'sinx + B'cosx = x.$

Now, $A = -\int x\,sinx\,dx = \int xd(cosx)$

$\quad = x\,cosx - \int cosxdx$

$\quad = xcosx - sinx + c_1$

and $B = \int x\,cosx\,dx = \int xd(sinx)$

$\quad = x\,sinx - \int sinx\,dx$

$\quad = x\,sinx + cosx + c_2$

Therefore, the required general solution is given by,

$y = Au + Bv$

$\quad = (x\,cosx - sinx + c_1)cosx + (x\,sinx + cosx + c_2)sinx$

$\quad = x + c_1\,cosx + c_2 sinx.$

Example 3:

Solve the equations $\frac{dy}{dx} + xsin2y = x^3cos^2y$, by the method of variation of parameters.

Solution:

The method of variation of parameters can be applied to solve only a linear differential equation. The given equation is not linear. We shall convert the given equation into a linear equation and then apply the method of variation of parameters. Dividing the given equation by cos^2y, we get,

$$sec^2y\frac{dy}{dx} + 2xtany = x^3 \qquad (8.14)$$

Putting $tany = z$, equation 8.14 becomes

$$\frac{dz}{dx} + 2xz = x^3, which\ is\ linear \qquad (8.15)$$

The homogeneous equation corresponding to equation 8.10 is

$$\frac{dz}{dx} + 2xz = 0\ or\ \frac{dz}{z} + 2xdx = 0 \qquad (8.16)$$

Integrating, we get $logz = logc - x^2$

i.e. the solution of equation 8.16 is

$$z = ce^{-x^2} \qquad (8.17)$$

Treating c as a function of x and differentiating equation 8.17 with respect to x,

$$\frac{dz}{dx} = -2cxe^{-x^2} + c'e^{-x^2} \qquad (8.18)$$

Using equation 8.18 and 8.17 in 8.15,

$$-2cxe^{-x^2} + c'e^{-x^2} + 2cxe^{-x^2} = x^2$$

$$c' = x^3e^{x^2}$$

$$c = \int x^3e^{x^2}dx + k$$

$$= \frac{1}{2}\int te^t dt + k, on\ \ putting\ x^2 = t$$
$$= \frac{1}{2}(te^t - e^t) + k$$
$$= \frac{1}{2}(x^2 - 1)e^{x^2} + k \qquad (8.19)$$

Using equation 8.19 in 8.17, the required general solution of equation 8.15 is

$$z = \frac{1}{2}(x^2 - 1) + ke^{-x^2}$$

Therefore the general solution of the given equation is

$$tan\, y = \frac{1}{2}(x^2 - 1) + ke^{-x^2}$$

where k is an arbitrary constant.

Example 4:

Apply the method of variation of parameters to solve

$$x^2y'' + 4xy' + 2y = e^x.$$

Solution:

Consider $x^2y'' + 4xy' + 2y = e^x$.

This is a homogeneous equation of second order.

Put $z = \log x$ and $\frac{d}{dz} = \theta$.

Hence $x\frac{dy}{dx} = \theta y$ and $x^2\frac{d^2y}{dx^2} = \theta(\theta - 1)y$

Equation (1) reduces to $\theta(\theta - 1)y + 4\theta y + 2y = 0$.

(i.e) $(\theta^2 + 3\theta + 2)y = 0$.

Its solution (C.F) is $ae^{-2x} + be^{-x}$ where a and b are constants.

The C.F of the given differential equation is $ae^{-2} + be^{-1}$

Let $u = x^{-2}$ and $v = x^{-1}$

Therefore, the general solution is given by

$$y = Au + Bv = Ax^{-2} + Bx^{-1}$$

Where A and B are functions of x determined by the equations

$$A'u + B'v = 0$$
$$A'u' + B'v' = R$$

Solving for A' and B' we get

$A' = -xe^x$ and $B' = e^x$

$A = \int(-xe^x)dx = -\int xe^x dx \quad = -[xe^x - e^x] + c_1 = \int e^x dx = e^x + c_2.$

The solution is,

$y = Au + Bv$

$= (-xe^x + e^x + c_1)x^{-2} + (e^x + c_2)x^{-1}$

$= -x^{-1}e^x + x^{-2}e^x + c_1x^{-2} + x^{-1}e^x + c_2x^{-1}.$

$\therefore y = c_1x^{-2} + c_2x^{-1} + x^{-2}e^x.$

Exercise

(i) $\frac{dy}{dx} - y\, tanx \ = e^x\, secx$

(ii) $x\frac{dy}{dx} + (1 + x)y = e^{-x}$

(iii) $\frac{d^2y}{dx^2} + y = x\, cosx$

(iv) $\frac{d^2y}{dx^2} + y = x\, sinx$

8.2 Total differential equations

The equation

$$Pdx + Qdy + Rdz = 0 \quad (8.20)$$

where P, Q and R are functions of x, y, z is called a total differential equation. If there exists a function u such that $du = Pdx + Qdy + Rdz$ then $u = c$ is a solution of equation 8.20 and in this case we say that equation 8.20 is integrable.

8.3 Methods of solving the differential equation $\boldsymbol{Pdx + Qdy + Rdz = 0}$

Method 1 (By taking one variable constant)

Suppose the condition of integrability is satisfied. Consider any one of the variables z as constant. Hence $dz = 0$. Now integrate the resulting equation $Pdx + Qdy = 0$ where the arbitrary constant of integration is taken as an arbitrary function of z. On differentiation of this integral with respect to x, y and z and comparing it with the given equation we can determine the arbitrary function of z.

Method 2 (P,Q, R are homogeneous functions)

If P, Q, R are homogeneous in x, y, z then the given equation can be solved by effecting the substitution $x = zu$ and $y = zv$.

Method 3(By forming auxiliary equations)

Consider $Pdx + Qdy + Rdz = 0$ which satisfies the condition of integrability

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

Comparing these two equations we have

$$\frac{dx}{\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)} = \frac{dy}{\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)} = \frac{dz}{\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}$$

These equations are called auxiliary equations and can be solved like simultaneous equation. If $u = a$ and $v = b$ are two solutions of the auxiliary equations then by comparing $adu + bdv = 0$ with the given equation we get the values of a and b and get the complete solution.

8.4 Solved problems

1. Solve $z^2dx + (z^2 - 2yz)dy + (2y^2 - y\ z - xz)dz = 0$

We can easily verify that the conditions of integrability is satisfied since the given equation is homogeneous we put $x = uz$ and $y = vz$ such that $dx = zdu + udz$ and $dy = zdv + vdz$.

Hence the given equation becomes

$$z^2(zdu + udz) + z^2(1 - 2v)(zdv + vdz) + (2v^2 - v - u)z^2dz = 0$$

$$\therefore z^3du + z^3(1 - 2v)dv + [u + v(1 - 2v) + 2v^3 - v - u]z^2dz = 0$$

$$\therefore z^3du + z^3(1 - 2v)dv = 0$$

$\therefore du + (1 - 2v)dv = 0$
$\therefore u + v - v^2 = c$
$\therefore \frac{x}{z} + \frac{y}{z} - (\frac{y}{z})^2 = c$
$\therefore (x + y)z - y^2 = cz^2$ *is the solution* .

2. Solve $yz^2(x^2 - yz)dx + zx^2(y^2 - xz)dy + xy^2(z^2 - xy)dz = 0$
Dividing by $x^2y^2z^2$ and regrouping we get

$$\frac{ydx - xdy}{y^2} + \frac{ydz - zdy}{z^2} + \frac{zdx - xdz}{x^2} = 0$$
$$\therefore d(\frac{x}{y}) + d(\frac{y}{z}) + d(\frac{z}{x}) = 0$$

Hence the required solution is $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 0.$

3. Verify the condition of integrability of
$(2xz - yz)dx + (2yz - zx)dy - (x^2 - zx + y^2)dz = 0$ and solve
Here $P = 2xz - yz$; $Q = 2yz - zx$; $R = -(x^2 - xy + y^2)$

$$\frac{\partial P}{\partial y} = -z;\ \frac{\partial P}{\partial z} = 2x - y;\ \frac{\partial Q}{\partial x} = -z;\ \frac{\partial Q}{\partial z} = 2y - z$$
$$\frac{\partial R}{\partial x} = -2x + y;\ \frac{\partial R}{\partial y} = x - 2y$$
$$\therefore P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$
$$(2xz - yz)(2y - x - x + 2y) + (2yz - zx)(y - 2x - 2x + y)$$
$$- (x^2 - xy + y^2)(-z + z) = 0$$

Treating $z =$ constant we have $dz = 0$
$\therefore$ The given equation becomes
$$(2xz - yz)dx + (2yz - zx)dy = 0.$$
(i.e) $(2x - y)dx + (2y - x)dy = 0$
On integration, $x^2 - xy + y^2 = \phi(x)$
Where $\phi(z)$ is an arbitrary function of z to be determined.
Differentiationg (1) we get
$$(2xz - yz)dx + (2y - x)dy - \phi'(z)dz = 0.$$
(i.e) $(2xz - yz)dx + (2yz - xz)dy - z\phi'(z)dz = 0.$
Comparing this with the given equation we get
$$z\phi'(z) = (x^2 - xz + y^2) = \phi(z)$$
$$\therefore z\frac{d\phi}{dz} = \phi(z)$$
(i.e) $\frac{d\phi}{\phi} = \frac{dz}{z}$
$\therefore \log \phi = \log cz$. Hence $\phi = cz$.
$\therefore$ The solution is $x^2 - xy + y^2 = cz$.

4. Solve $z(z - y)dx + (z + x)zdy + x(x + y)dz = 0$ by forming the auxillary
equations.
Here $P = z(z - y); Q = (z + x)z; R = x(x + y)$.

$$\frac{\partial P}{\partial y} = -z;\ \frac{\partial P}{\partial y} = 2z - z;\ \frac{\partial Q}{\partial x} = 2z + x;\ \frac{\partial R}{\partial x} = 2x + y;\ \frac{\partial R}{\partial y} = x$$

We can easily verify that the condition of integrability is satisfied.
Now, the auxiliary equation are

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}$$

$$\therefore \frac{dx}{2z + x - x} = \frac{dy}{2x + 2y - 2z} = \frac{dz}{-z - z}$$

(i.e) $\frac{dx}{z} = \frac{dy}{x+y-z} = \frac{dz}{-z}$

Taking the first and third ratio we get $dx + dz = 0$

Hence $x + z = u$.

Also we have $\frac{dx+dy}{x+y} = \frac{dz}{-z}$

Hence $\log(x + y) + \log z = \log v$

$\therefore\ (x + y)z = v$.

Now, $adu + bdv = 0$.

$$\Rightarrow a(dx + dz) + b[zdx + zdy + (x + y)dz] = 0$$

$$\therefore (a + bz)dx + bzdy + (a + x + y)dz = 0.$$

Comparing it with the given equation we get

$$a + bz = z(z - y); bz = z(z + x); a + x + y = x(x + y).$$

Hence $b = z + x = u$ and

$$a = z(z - y) - bz = z(z - y) - (z + x)z = -z(x + y) = -v$$

$$\therefore a\,du + bdv = 0\ \Rightarrow -vdu + udv = 0.$$

On integration $\frac{v}{u} = c$. Hence $v = cu$.

$\therefore (x + y)z = c(x + z)$ is the solution.

5. Solve $yz^2(x^2 - yz)dx + zx^2(y^2 - xz)dy + xy^2(z^2 - xy)dz = 0$.

Dividing by $x^2\ y^2\ z^2$ and regrouping we get,

$$\frac{ydx - xdy}{y^2} + \frac{ydz - zdy}{z^2} + \frac{zdx - xdz}{x^2} = 0.$$

$\therefore\ d\left(\frac{x}{y}\right) + d\left(\frac{y}{z}\right) + d\left(\frac{z}{x}\right) = 0$

Hence the required solution is $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = c$.

Exercise

Solve:

(i) $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$

(ii) $(y + z)dx + (z + x)dy + (x + y)dz = 0$

(iii) $(x - y)dx - xdy + zdz = 0$

(iv) $(yz + z^2)dx - xzdy + xydz = 0$.

BLOCK - III PARTIAL DIFFERENTIAL EQUATIONS

UNIT- IX NECESSARY AND SUFFICIENT CONDITION OF INTEGRABILITY OF PDX+QDY+RDZ=0

Structure:

9.1 NECESSARY AND SUFFICIENT CONDITION OF INTEGRABILITY

In a total differential equation, we have the differential coefficients of several dependent variables with reference to a single independent variable. Such an equation in three variables is represented by

$$Pdx + Qdy + Rdz = 0 \qquad (9.1.1)$$

Criterion of integrability

Let (9.1.1) have an integral

$$u(x, y, z) = a \qquad (9.1.2)$$

where a is an arbitrary constant.

Then differentiating (9.1.2) totally,

$$\frac{\delta u}{\delta x} dx + \frac{\delta u}{\delta y} dy + \frac{\delta u}{\delta z} dz = 0.$$

This is identical with (9.1.1). Hence,

$$\frac{\delta u}{\delta x} = \mu P, \quad \frac{\delta u}{\delta y} = \mu Q \text{ and } \frac{\delta u}{\delta z} = \mu R.$$

$$\frac{\delta}{\delta y}(\mu P) = \frac{\delta^2 u}{\delta y \delta x} = \frac{\delta^2 u}{\delta x \delta y} = \frac{\delta}{\delta x}(\mu Q).$$

$$\mu\left(\frac{\delta P}{\delta y} - \frac{\delta Q}{\delta x}\right) = Q\frac{\delta \mu}{\delta x} - P\frac{\delta \mu}{\delta y}$$

Similarly,

$$\mu\left(\frac{\delta Q}{\delta z} - \frac{\delta R}{\delta y}\right) = R\frac{\delta \mu}{\delta y} - Q\frac{\delta \mu}{\delta x}$$

and

$$\mu\left(\frac{\delta R}{\delta x} - \frac{\delta P}{\delta z}\right) = P\frac{\delta \mu}{\delta y} - R\frac{\delta \mu}{\delta x}$$

Multiplying (i), (ii), (iii) by R, P and Q respectively and adding, we obtain

$$P\left(\frac{\delta Q}{\delta z} - \frac{\delta R}{\delta y}\right) + Q\left(\frac{\delta R}{\delta x} - \frac{\delta P}{\delta z}\right) + R\left(\frac{\delta P}{\delta y} - \frac{\delta Q}{\delta x}\right) = 0. \qquad (I)$$

The above condition is apoken of as the condition of integrability of equation (9.1.1).

We have just shown that the above condition (I) is necessary for the existence of the integral of equation (9.1.1).

We shall show that the condition is *sufficient* by proving that an integral of (9.1.1) can be found when (I) holds good.

1.1 If, relation (I) exists among P, Q, R a similar relation holds good between the coefficients of

$$\mu P dx + \mu Q dy + \mu R dz = 0 \qquad \text{(II)}$$

where μ is any function of x, y, z.

If $Pdx + Qdy$ (where z is a parameter) is not an exact differential, we can always find an integration factor μ such that $\mu Pdx + \mu Qdy$ is exact. Hence, without loss of generality, $Pdx + Qdy$ can be regarded as an exact differential and (II) as the equation to be considered.

$$\therefore \frac{\delta P}{\delta y} = \frac{\delta Q}{\delta x}$$

Let

$$V = \int (Pdx + Qdy).$$

i.e.,

$$\frac{\delta V}{\delta x} dx + \frac{\delta V}{\delta y} dy = Pdx + Qdy$$

Hence

$$\frac{\delta V}{\delta x} = P \text{and} \frac{\delta V}{\delta x} = Q$$

$$\frac{\delta P}{\delta z} = \frac{\delta^2 V}{\delta z \delta x} \text{and} \frac{\delta Q}{\delta z} = \frac{\delta^2 V}{\delta z \delta y}$$

(I) now reduces to

$$\frac{\delta V}{\delta x}\left(\frac{\delta^2 V}{\delta z \delta y} - \frac{\delta R}{\delta y}\right) + \frac{\delta V}{\delta y}\left(\frac{\delta R}{\delta x} - \frac{\delta^2 V}{\delta z \delta x}\right) = 0$$

$$i.e., \begin{vmatrix} \frac{\delta V}{\delta x} & \frac{\delta}{\delta x}\left(\frac{\delta V}{\delta z} - R\right) \\ \frac{\delta V}{\delta y} & \frac{\delta}{\delta y}\left(\frac{\delta V}{\delta z} - R\right) \end{vmatrix} = 0$$

$$i.e., \frac{\delta(V, \frac{\delta V}{\delta z} - R)}{\delta(x,y)} = 0.$$

$\therefore$ A relation independent of x and y exists between V and $\frac{\delta V}{\delta z} - R$; hence $\frac{\delta V}{\delta z} - R$ can be expressed as a function of z and V.

Let $\frac{\delta V}{\delta z} - R = \phi(z, V)$.

Since $Pdx + Qdy + Rdz \quad = \frac{\delta V}{\delta x}dx + \frac{\delta V}{\delta y}dy + \frac{\delta V}{\delta z}dz + \left(R - \frac{\delta V}{\delta z}\right)dz$

$= dV - \phi(z, V)dz.$

This, being an equation in two variables z and V, leads on integration to $F(V, z) = 0$.

Hence (I) is the necessary and sufficient criterion for integrability of (9.1.1).

9.2 Rules for integrating Pdx+Qdy+Rdz=0

• When the condition of integrability is satisfied, consider one of the variables, say z, as constant, i.e., $dz = 0$ and integrate the equation $Pdx + Qdy = 0$.

• Put the arbitrary constant of integration that occurs in this integral as an arbitrary function of z.

• This is justified as the arbitrary constant in this integral is a constant only with respect to x and y.

• Differentiating this integral with respect to x, y and z and comparing the result with (9.1.1), we can determine the arbitrary function of z.

The following examples will illustrate the process.

9.3 Examples

Ex.1. Solve

$$(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0 \qquad (9.3.1)$$

Here the condition of integrability is satisfied.

Neglecting$(y^2 - xy)dz$, we get

$$(y^2 + yz)dx + (xz + z^2)dy = 0$$

$$\frac{dx}{x+z} + \frac{zdy}{y(y+z)} = 0.$$

Integrating, on the assumption that z is a constant,

$$log(x+z) + log\frac{y}{y+z} = logc$$

$$i.e., \frac{(x+z)y}{y+z} = c = f(z), \qquad (9.3.2)$$

where the arbitrary constant c of integration is put as $f(z)$, an arbitrary function of z.

Differentiating (9.3.2) totally with respect to $x, y, z,$

$$\frac{(y+z)[(dx+dz)y+(x+z)dy]-y(x+z)(dy+dz)}{(y+z)^2}=f'(z)dz$$

$$i.e., (y^2+yz)dx+(xz+z^2)dy+dz[y^2-xy-f'(z)(y+z)^2]=0.$$

Comparing this with (9.3.1), we have $f'(z)(y+z)^2dz=0$.
As $(y+z)^2dz\neq 0$, $f'(z)=0\ldots f(z)=$ constant c.
Hence the integral of (9.3.1) is $y(x+z)=c(y+z)$.

Exercises

Verify the condition of integrability in the following equations and solve them:

1. $(y+z)dx+(z+x)dy+(x+y)dz=0.$

2. $(y+z)dx+dy+dz=0.$

UNIT-X PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ODER

Structure

10.1 Partial Differential Equations of the First Order

We now consider equations in which the number of independent variables is two or more and only one dependent variable. We usally denote this by z and the independent variables by x and y, if there be two; if there be n independent variables, we shall call them $x_1, x_2, \dots x_n$. The partial derivatives $\frac{\delta z}{\delta x}, \frac{\delta z}{\delta y}$ are denoted by p and q while, in the latter case, $\frac{\delta z}{\delta x_1}, \frac{\delta z}{\delta x_2} \dots \frac{\delta z}{\delta x_n}$ are represented by $p_1, p_2, \dots p_n$ respectively.

Partial differential equations are those which involve oe or more partial derivatives. The order of a partial differential equation is determined by the highest order of the partial derivative occuring in it. We consider only partial differential equations of the first order.

10.2 Classification of Integral

Let the partial differential equation be

$$F(x, y, z, p, q) = 0 \tag{10.2.1}$$

Let the solution of this be

$$\phi(x, y, z, a, b) = 0 \tag{10.2.2}$$

where a and b are arbitrary constants.

The solution (10.2.2) which contains as many arbitrary constants as there are independent variables is called the complete integral of (10.2.1).

A particular integral of (10.2.1) is that got by giving particular values to a and b in (10.2.2).

10.3 Singular Integral

The elimation of a and b between

$$\phi(x, y, z, a, b) = 0; \frac{\delta \phi}{\delta a} = 0 \text{ and } \frac{\delta \phi}{\delta b} = 0$$

when it exists, is called the singular integral.

Geometrically, this includes the envelope of the two parameter surfaces represented by the complete integral (10.2.2) of (10.2.1). The two parameters occuring in (10.2.2) are a and b.

The locus of all points whose coordinates with the corresponding values of p and q satisfy (10.2.1) is the doubly infinite system of surfaces represented by (10.2.2). As the envelope of these surfaces touches at each point one member of the system (10.2.2) the coordinates of any point of the envelope and the associated p and q satisfy (10.2.1) and is thus a solution of (10.2.1). This is a singular solution as we cannot deduce thus from (10.2.2) by giving any values of a and b.

In any actual example, one should test whether what is apparently a singular integral really satisfies the differential equation.

10.4 General integral

In (10.2.2), we shall assume an arbitrary relation between a and b of the form $b = f(a)$.

Then (10.2.2) becomes

$$\phi[x, y, z, a, f(a)] = 0$$

Differentiating this partially with respect to a,

$$\frac{\delta\phi}{\delta a} + \frac{\delta\phi}{\delta b} f'(a) = 0$$

The eliminant of a between these two equations is called the general integral of (10.2.1).

The above two equations represent a curve, viz., the curves of intersection of two consecutive surfaces of the system $\phi[x, y, z, a, f(a)] = 0$ for a particularvalues of a. The envelope of the family of the surfaces touches them along this curve, which is called the characteristic of the envelope. Thus the gerenal integral represents the envelope of a family of surfaces, considered as composed of its characteristics.

UNIT-XI DERIVATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Structure

11.1 Derivation of Partial Differential Equations:

Partial differential equations can be derived either by the elimination of (i) arbitrary constants from a relation between x,y,z or (ii) of arbitrary function of these variables.

11.2 By Elimination of Constants

Let, $\phi(x, y, z, a, b) = 0$ (11.2.1)

be a relation between x, y, z involving two arbitrary constans a and b.

Differentiating $(11.2.1)$ with respect to x and y partially, we get,

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p = 0$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} q = 0$$

Eliminating a and b, we have a partial differential equation of the first order of the form, $F(x, y, z, p, q) = 0$

Here the number of constants to be eliminated is equal to the number of independent variables and an equation of the first order results. If the number of constants to be eliminated is greater than the number of independent variables, equations of the second and higher derivatives are deduced.

Example: Eliminate a and b from

$$z = (x + a)(y + b)$$

Differentiating with respect to x and y partially,

$$p = y + b$$

and

$$q = x + a$$

Eliminating a and b , $z = pq$.

Excercise: Eliminate a and b from $z = ax + by + a$

11.3 By Elimination of an Arbitrary Function

Let u and v be any two functions of x,y,z an be connected by an arbitrary relation $\phi(u,v)=0\cdots(1a)$

By eliminating ϕ, we shall form a partial differential equation and show that this is linear, i.e., of the first degree in p and q.

Differentiating (1a) partially with respect to x and y,

$$\frac{\partial\phi}{\partial u}(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p)+\frac{\partial\phi}{\partial v}(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p)=0$$

$$\frac{\partial\phi}{\partial u}(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q)+\frac{\partial\phi}{\partial v}(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q)=0$$

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$, we have

$$(u_x+u_zp)(v_y+v_zq)=(u_y+u_zq)(v_x+v_zp)$$

where $u_x=\frac{\partial u}{\partial x}$, $u_y=\frac{\partial u}{\partial y}$, etc. This equation can be put in the form

$Pp+Qq=R$ where

$$P=u_yv_z-u_zv_y, Q=u_zv_x-u_xv_z, \text{ and}$$
$$R=u_xv_y-u_yv_x$$

Example: Eliminate the arbitrary function from

$$z=f(x^2+y^2)$$

Differentiating partially with respect to x and y,

$$p=f'(x^2+y^2)2x \quad \text{and} \quad q=f'(x^2+y^2)2y.$$

Eliminating $f'(x^2+y^2)$ between latter two equations

$$py=qx$$

Exercise: Eliminate the arbitrary function $z=e^yf(x+y)$ Consider the equations $u=a$ and $v=b$, where a and b area arbitrary constants. By eliminating a and b , we form the differential equations corresponding to them.

11.4 Special Methods; Standard forms

11.4.1 Standard 1

Equations, in which the variables do not occur explicitly, can be written in the form $F(p,q)=0$.

A solution of this is $z=ax+by+c$, where a and b are connected by $F(a,b)=0$. Solving this for b, $b=f(a)$.

Hence the complete integral is,

$$z=ax+yf(a)+c$$

The singular integral is obtained by eliminating a and c between

$$z=ax+yf(a)+c$$

$$0=x+yf'(a)$$

$$0=1$$

The last equation is absurd and shows that there is not singular integral in the case.

To obtain the general integral, we assume an arbitrary relation $c = \phi(a)$. Then

$$z = ax + yf(a) + \phi(a)$$

Differentiating partially with respect to a,

$$0 = x + yf'(a) + \phi'(a)$$

The elimination of a between these equations is the general integral.

Note:

The singular and general integrals must be indicated in every equation besides the complete integral. Then only the equation besides is said to be the completely solved.

Example: $p^2 + q^2 = npq$

Let the solution be $z = ax + by + c$, where

$$a^2 + b^2 = nab.$$

Solving,

$$b = \frac{a(n \pm \sqrt{n^2-4})}{2}$$

The complete integral is

$$z = ax + \frac{ya}{2}(n \pm \sqrt{n^2 - 4}) + c$$

Differentiating partially with respect to c, we find that there is no singular integral as we get an absurd result.

To find the general integral, put $c = f(a)$.

$$z = ax + \frac{ay}{2}(n + \sqrt{n^2 - 4}) + f(a).$$

Differentiating partially with respect to a,

$$0 = x + \frac{y}{2}(n + \sqrt{n^2 - 4}) + f'(a)$$

The elimination of a between the equation is the general integral.

Exercise:

Solve the following: (i) $z = 3p^2 - 2q^2 = 4pq$ (ii) $p^2 + q^2 = 4$

11.4.2 Standard II

Only one of the variables x,y, z occurs explicitly. Such equations can be written in one of the forms,

$$F(x, p, q) = 0; \quad F(y, p, q) = 0; \quad F(z, p, q) = 0$$

- Let us consider the form $F(x, p, q) = 0$.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$
$$= pdx + qdy$$

Let us assume that $q = a$.

The equation becomes $F(x, p, a) = 0$.

Solving this for p, we get $p = \phi(x, a)$.

$$\therefore dz = \phi(x, a)dx + ady.$$

$$\therefore z = \int \phi(x,a)dx + ay + b.$$

This contains two arbitrary constants a and b and hence it is a complete integral.

• Let us consider the form $F(y,p,q) = 0$.

Let us assume that $p = a$.

$$\therefore F(y,a,q) = 0.$$

$$\therefore q = \phi(y,a).$$

Hence $dz = adx + \phi(y,a)dy$.

$$\therefore z = ax + \int \phi(y,a)dy + b$$

which is a complete integral.

• Let us consider the equation $F(z,p,q) = 0$.

Let us assume that $q = ap$.

Then this equation becomes $F(z,p,ap) = 0$

$$\text{i.e., } p = \phi(z,a)$$

Hence $dz = \phi(z,a)dx + a\phi(z,a)dy$

$$\text{i.e., } \frac{dz}{\phi(z,a)} = dx + ady$$

$$\text{i.e., } \int \frac{dz}{\phi(z,a)} = x + ay + b \quad \text{which is a complete integral.}$$

Examples:

$Solve$: $(i) q = xp + p^2$. $(ii) p = y^2q^2$ $(iii) p(1 + q^2) = q(z - 1)$

(i) $q = xp + p^2$

Let $q = a$. Then $a = xp + p^2$ i.e., $p^2 + xp - a = 0$

$$\therefore p = \frac{-x \pm \sqrt{x^2+4a}}{2}$$

$$\text{Hence} dz = \frac{-x \pm \sqrt{x^2+4a}}{x} dx + ady$$

$$\therefore z = \int \frac{-x \pm \sqrt{x^2+4a}}{2} dx + ay + b$$

$$= -\frac{x^2}{4} \pm \{\frac{x}{4}\sqrt{4a + x^2} + asinh^{-1}(\frac{x}{2\sqrt{a}})\} + ay + b.$$

(ii) $p = y^2q^2$. Let $p = a^2$.Then $q = \pm(\frac{a}{y})$

$$\text{Hence} dz = a^2dx \pm (\frac{a}{y})dy.$$

$$z^2 = a^2x \pm alogy + b.$$

(iii) $p(1+q^2) = q(z-1)$.

Let $q = ap$. Then $p(1+a^2p^2) = ap(z-1)$

.

$$\therefore 1 + a^2p^2 = a(z-1).$$

$$\text{(i.e.,) } p = \pm\frac{\sqrt{(az-a-1)}}{a}.$$

$$\text{Hence } dz = \pm\frac{\sqrt{(az-a-1)}}{a}dx \pm a\frac{\sqrt{(az-a-1)}}{a}dy.$$

$$\text{(ie.,) } \pm\frac{\text{a dz}}{\sqrt{(az-a-1)}} = dx + ady.$$

$$\text{(ie.,) } \pm\int\frac{\text{a dz}}{\sqrt{(az-a-1)}} = x + ay + b.$$

$$\text{(ie.,) } \pm 2\sqrt{(az-a-1)} = x + 2y + b.$$

Exercise:

$Solve$: $(i) z(p^2+q^2+1) = a^2$ $(ii) p = 2qx$ $(iii) p^2+q^2 = z$

11.4.4 Standard III

Equations of the form $f_1(x,p) = f_2(y,q)$. In this form the equation is of the first order and the variables are separable. In this equation z does not appear. We shall assume a tentative solution that each of these quantities is equal to a.

Solving $f_1(x,p) = a, p = \phi_1(a,x)$.
Solving $f_2(y,q) = a, q = \phi_2(a,y)$.
Hence $dz = \phi_1(a,x)dx + \phi_2(a,y)dy$.

$$\therefore z = \int \phi_1(a,x)dx + \int \phi_2(a,y)dy + b$$

which is the complete integral.

Example: Solve the equation $p + q = x + y$.

We can write the equation in the form $p - x = y - q$.
Let $p - x = a$. Then $y - q = a$.
Hence $p = x + a$, $q = y - a$.

$$\therefore dz = (x+a)dx + (y-a)dy.$$

$$\therefore z = \frac{(x+a)^2}{x} + \frac{(y-a)^2}{2} + b.$$

There is no singular integral and the general integral is found asusual.

Exercise: (i) $p^2 + q^2 = x + y$ $(ii) p^2 + q^2 = x^2 + y^2$

11.4.5 Standard IV; Clairant's form.

This is of the form $z = px + qy + f(p, q)$.

The solution of the equation is $z = ax + by + f(a, b)$ for $p = a$ and $q = b$ can easily be verified to satisfy the given equation.

Example: Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$.

The complete integral is obviously

$$z = ax + by + \sqrt{1 + a^2 + b^2}.$$

To find the singular integral, differentiating partially with respect to a and b.

$$x + \frac{a}{\sqrt{1+a^2+b^2}} = 0.$$

$$\text{and} \quad y + \frac{b}{\sqrt{1+a^2+b^2}} = 0.$$

Eliminating a and b the singular integral is

$$x^2 + y^2 + z^2 = 1.$$

To find the general integral, assume $b = f(a)$ where f is arbitrary.

Differentiating partially with respect to a and eliminate the a between two equations.

Exercise: $(i) z = px + qy + pq \quad (ii) z = px + qy + 2\sqrt{pq}$.

<u>BLOCK- XII:</u>

STANDARD FORMS OF PARTIAL DIFFERENTIAL EQUATIONS AND TRAJECTORIES

UNIT-XII STANDARD FORMS OF PARTIAL DIFFERENTIAL EQUATIONS

Structure

12.1 Standard Form of Partial Differential Equations(PDE)

In generally, a PDE having three kinds of standard forms.

12.1.1 TYPE 1 :

$$\boldsymbol{If\ the\ equation\ of\ the\ form\ f(p,q)=0} \qquad \textbf{(12.1.1)}$$

Clearly, $\boldsymbol{z = ax + by + c}$ where a,b are such that $f(a,b) = 0$ satisfies (12.1.1).

Now, solving for b from $f(p,q) = 0$, we get $b = ga$.

The complete integral is

$$\boldsymbol{z = ax + yg(a) + c} \qquad (12.1.2)$$

To find the general solution, put $c = \varphi(a)$ in (12.1.2).

$$we\ get,\ z = ax + yg(a) + \varphi(a) \qquad (12.1.3)$$

$Differentiating$ (12.1.3) $w.r.to\ a$

$$we\ get,\ x + yg'(a) + \varphi'(a) = 0. \quad (12.1.4)$$

Eliminating a from (12.1.3) and (12.1.4) we get the general solution. To find the singular integral we have to eliminate $a,\ c$ from the following three equations.

$$z = ax + g(a)y + c;\ x + yg'(a) = 0; 0 = 1.$$

The above equation, $0 = 1$ assures that there is no singular integral in this case.

1.1 Example

Solve : $pq + p + q = 0$

Solution : $z = ax + by + c$

where, $ab + a + b = 0$ is a complete integral.

$$ab + a + b = 0 \Rightarrow b = -\frac{a}{a+1}$$

The complete integral is

$$z = ax - (\frac{a}{a+1})y + c \qquad (12.1.5)$$

There is no singular integral. To find the general integral put $c = \varphi(a)$ in (12.1.5).

$$\therefore We\ get, z = ax - (\frac{a}{a+1})y + \varphi(a) \qquad (12.1.6)$$

Differentiating w.r.to a we get

$$x - (\frac{1}{(a+1)^2})y + \varphi'(a)$$

Eliminating a from (12.1.5) & (12.1.6) we get the general solution.

12.1.2 TYPE 2 :

$\boldsymbol{If\ the\ equation\ of\ the\ form\ f(p,q) + qy + px = z}$ **(12.2.1)**

The complete solution is given by $z = ax + by + f(a,b)$.

The general and singular integrals are obtained by the usual method.

1.2 Example

Solve : $px + qy + (\frac{q}{p} - p) = 0$

Solution :

$$z = ax + by + (\frac{b}{a} - a) \qquad (12.2.2)$$

To find a singular integral we differentiate (61) w.r.to a, b we get,

$$x - (\frac{b}{a^2}) - 1 = 0. \qquad (12.2.3)$$

$$y + (\frac{1}{a}) = 0. \qquad (12.2.4)$$

(12.2.3) $\Rightarrow$ $a = -(\frac{1}{y})$ and (12.2.4) $\Rightarrow$ b= $\frac{(x-1)}{y^2}$

Substituting the values of a, b in (12.2.2) we get, $yz = 1 - x$. is the singular solution.

To find general integral put $b = \varphi(a)$ in (12.2.2) and we get

$$z = ax + \varphi(a)y + \frac{\varphi(a)}{a} - a \qquad (12.2.5)$$

Differentiating w.r.to a we get,

$$x + \varphi'(a)y - \frac{\varphi(a)}{a^2} + \frac{\varphi'(a)}{a} - 1 = 0 \qquad (12.2.6)$$

Eliminating a from (12.2.5) and (12.2.6) we get the general solution.

12.1.3 TYPE 3 :

$$\boldsymbol{If\ the\ equation\ of\ the\ form\ f(z,p,q) = 0} \qquad \textbf{(12.3.1)}$$

Put $z = F(x + ay) = F(u)$ where a is a constant.

$$p = \frac{dz}{du}\frac{\partial u}{\partial x}; q = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting the values of p, q in (12.3.1) we get

$$f(z, \frac{dz}{du}, a\frac{dz}{du}) = 0 \qquad (12.3.2)$$

Which is an ordinary differential equation of order.

A solution of (12.3.1) gives an expression of the form $\frac{dz}{du} = \varphi(z, a)$ $\therefore \frac{dz}{\varphi(z,a)} = du.$

Hence, it's solution is given by $g(z, a) = u + b = x + ay + b$ is the complete integral of (12.3.1).

The general and singular integrals are found by the usual methods.

1.3 Example

Solve : $4(1 + z^3) = 9z^4pq$

Solution :

Put $z = F(x + ay) = F(u)$ where a is a constant.

$$p = \frac{dz}{du}\frac{\partial u}{\partial x}; q = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}; 4(1 + z^3) = 9z^4(\frac{dz}{du})^2.$$

$$\therefore \frac{dz}{du} = \frac{2}{\sqrt{a}}(\frac{\sqrt{(1+z^3)}}{3z^2}). Hence, \frac{3z^2\sqrt{a}}{\sqrt{(1+z^3)}}dz = 2du.$$

Integrating, $\sqrt{a}\sqrt{(1 + z^2)} = u + b.$

Hence, $a(1 + z^3) = (u + b)^2.$

$\therefore$ The complete integral is $a(1 + z^3) = (x + ay + b)^2.$

Differentiating the complete integral w.r.to $a, b.$

we get $1 + z^3 = 2(x + ay + b)y$ and $2(x + ay + b) = 0.$

Hence, $1 + z^3 = 0$ which is the singular solution.

12.1.4 TYPE 4 :

$$\boldsymbol{If\ the\ equation\ of\ the\ form\ f_1(x,p) = f_2(y,q)} \quad \textbf{(12.4.1)}$$

Put $f_1(x,p) = f_2(y,q) = a$ where, a is a constant. Solving this equation (12.3.3) w.r.to p,q we get, $p = g_1(x,a)$ and $q = g_2(y,a)$.

We know that $dz = pdx + qdy$. $\Rightarrow 4$ $dz = g_1(x,a)dx + g_2(y,a)dy$.

Hence,

$$z = \int g_1(x,a)dx + \int g_2(y,a)dy + b$$

is the complete integral.

The general and singular solutions are found as usual.

1.4 Example

Solve : $qe^x = pe^y$.

Solution :

The given equation can be written as $pe^{-x} = qe^{-y}$.

$pe^{-x} = qe^{-y} = a \Rightarrow p = ae^x$., $q = ae^y$.

$\therefore\ z = a(e^x + e^y) + b$ is the complete integral.

We can't find easily the singular integral but as usual we found the general solution.

1.5 Excercises

Find the complete integral value for the following by using four types of standard forms.

1. $p + q = pq$
2. $z = px + qy + pq$
3. $p^3 + q^3 = 27z$
4. $p^2 + q^2 = x + y$

12.2 CHARPIT'S METHOD

We now give a general method of solving a partial differential equation(PDE) of first order due to Charpit.

$$\boldsymbol{Consider\ a\ partial\ differential\ equation\ f(x,y,z,p,q) = 0} \quad \textbf{(12.5.1)}$$

$$We\ know\ that\ dz = pdx + qdy \qquad (12.5.2)$$

If we find the relation between x, y, z, p, q then

$$F(x,y,z,p,q) = 0 \qquad (12.5.3)$$

Find p, q from (12.5.1) and (12.5.3) and substitute the values of p, q in (12.5.2).

We get the ordinary differential equation(ODE) and the solution of this ODE satisfies (12.5.1).

Next to find the relation of (12.5.3), differentiating (12.5.1) and (12.5.3) w.r.to x, y we get,

$$f_x + f_z p + f_p p_x + f_q q_x = 0 \quad (12.5.4)$$

$$F_x + F_z p + F_p p_x + F_q q_x = 0 \quad (12.5.5)$$

$$f_y + f_z q + f_p p_y + f_q q_y = 0 \quad (12.5.6)$$

$$F_y + F_z q + F_p p_y + F_q q_y = 0 \quad (12.5.7)$$

Eliminating p_x from (12.5.3), (12.5.4) we get,

$$(f_x + f_z p + f_q q_x)F_p - (F_x + F_z p + F_q q_x)f_p = 0$$

$$(ie.,)(f_x F_p - F_x f_p) + (f_z F_p - F_z f_p)p + (f_q F_p - F_q f_q)q_x = 0 \quad (12.5.8)$$

Similar way to eliminate (12.5.5), (12.5.6)

$$(ie.,)(f_y F_q - F_y f_q) + (f_z F_q - F_z f_q)q + (f_p F_q - F_p f_q)p_y = 0. \quad (12.5.9)$$

Since, $q_x = \frac{\partial^2 z}{\partial x \partial y} = p_y$ adding (12.5.7), (12.5.8) and rearranging we get,

$$(f_x + pf_z)F_p + (f_y + f_z q)F_q + (-pf_p - qf_q)F_z + (-f_p)F_x + (-f_q)F_y = 0 \quad (12.5.10)$$

This (12.5.10) is a linear PDE of first order and its integral is obtained by solving the auxillary equations

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad (12.5.11)$$

We can use (12.5.11) to obtain the function $F(x, y, z, p, q) = 0$

12.2.1 Solution Procedures for Charpit's Method

- Express the given equation as $f(x, y, z, p, q) = 0$ and take the partial derivatives of f_p, f_q, f_x, f_y, f_z
- Write the auxillary equation and obtain a simple relation involving atleast one of p or q.
- Use the given equation to get p, q.
- Follow the result $dz = pdx + qdy$ and integrate to found the complete integral, general integral solutions.

12.2.2 Examples

Problem 1. Find the complete integral for $z = px + qy + p^2 + q^2$

Solution:

$$Let\ f(x,y,z,p,q) = z - px - qy - p^2 - q^2 \qquad (12.5.12)$$

$$f_p = -x - 2p; f_q = -y - 2q; f_x = -p; f_y = -q; f_z = 1$$

The charpit's auxillary equations are,

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\therefore \frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p)} = \frac{dx}{x+2p} + \frac{dy}{y+2q}$$

Taking $\frac{dp}{0} = \frac{dq}{0}$ we get, $dp = 0 \Rightarrow p = a.$

Similarly, $dq = 0 \Rightarrow q = b.$ Substitute these values in (12.5.12).

We get, $z = ax + by + a^2 + b^2$ is the complete integral.

Problem 2. Find the complete integral for $z = px + qy + f(p, y)$

Solution :

The charpit's auxillary equations are given by

$$\frac{dp}{0} = \frac{dq}{0} = \cdots$$

$\therefore$ p = a and q = b.

Hence, the complete integral is $z = ax + by + f(a, b)$.

12.2.3 Excercises

Find the complete integral value for the following by using Charpit's Method

1. $q = (z + px)^2$
2. $p^2 = q + xp$
3. $p = (qy + z)^2$

UNIT-XIII FLOW OF WATER FROM AN ORIFICE

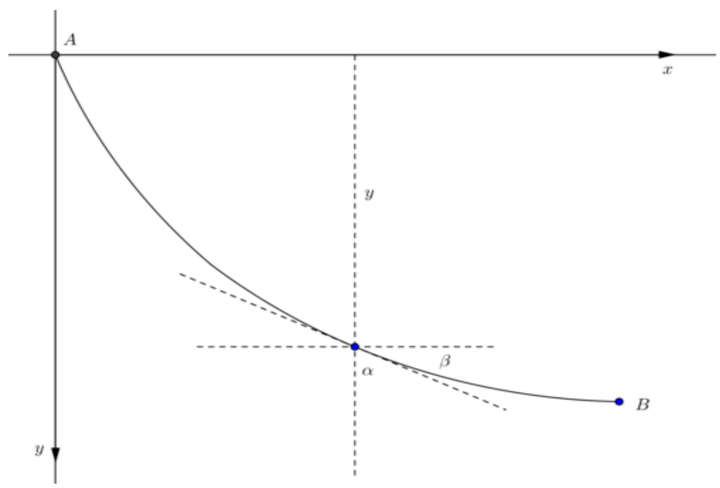
Structure

13.1 The Brachistochrone Problem

Theorem: If A, B are two point in a space but not in the same vertical lines. Let A, B be connected by a plane curve C and suppose A is higher level than B. Consider the smooth particle which falls from A to B under gravity along the curve C starting from rest. The problem of finding the shape of C which ensures that the time of fall from A to B is minimum is called the **Brachistochrone problem**.

Proof: We now give the Bernouli's solution to this problem. In Figure 1 we construct a horizontal and downward vertical lines through A as the x and y axes. Let $P(x, y)$ be the position of the particle at time t. Let s be the arc length AP. Let α be the angle made by the tangent at P with y axis.

Figure 1: Diagram representation of Brachistochrone problem



Let v be the velocity of the particle at P.

Taking the $x - axis$ as the standard level initially potential energy $V = 0$ and kinetic energy $T = 0$.

Hence, by the principle of conservation of energy is

$$\frac{1}{2}mv^2 - mgy = 0.$$

$$\therefore v = \sqrt{2gy} \qquad (13.1.1)$$

$$by\ Snell's\ Law, \frac{sin\alpha}{v} = c(constant) \quad (13.1.2)$$

Further,

$$y' = tan(90 - \alpha) = cot\alpha = \frac{cos\alpha}{sin\alpha}.$$

$$\therefore sin\alpha = \frac{cos\alpha}{y'} = \frac{1}{y'sec\alpha}.$$

$$sin\alpha = \frac{1}{y'\sqrt{1+tan^2\alpha}} = \frac{1}{y'\sqrt{1+(\frac{1}{y'})^2}} = \frac{1}{\sqrt{1+(y')^2}}$$

using (13.1.2)

$$cv^2 = sin^2\alpha = \frac{1}{1+(y')^2}$$

$$Using\ (13.1.1)\ and\ (13.1.2)\ we\ get,\ c(2gy) = \frac{1}{(1+(y')^2)}$$

$$\Rightarrow y(1 + (y')^2) = \frac{1}{2gc} = k(constant)$$

$$(ie.,)y(1 + (\frac{dy}{dx})^2) = k.$$

$$Seperating\ the\ variables\ we\ get, dx = \sqrt{(\frac{y}{k-y})}dy \quad (13.1.3)$$

$$Let \sqrt{(\frac{y}{k-y})} = tan\varphi \quad (13.1.4)$$

$$\therefore \frac{y}{k-y} = tan^2\varphi \Rightarrow y = (k-y)tan^2\varphi.$$

$$\therefore ysec^2\varphi = ktan^2\varphi \Rightarrow y = ksin^2\varphi.$$

$$Hence,\ dy = 2ksin\varphi cos\varphi d\varphi.$$

$$Now,\ (13.1.3) \Rightarrow dx = tan\varphi dy = 2ksin^2\varphi d\varphi.$$

$$\therefore dx = k(1 - cos2\varphi)d\varphi.$$

$$Integrating\ this\ we\ get, x = k(\varphi - \frac{1}{2}sin2\varphi) + c$$

$$x = \frac{k}{2}(2\varphi - sin2\varphi) + c \quad (13.1.5)$$

The curve passing through the origin $\therefore$ (13.1.4) $\Rightarrow \varphi = 0$ when $x = 0$ and hence (13.1.5) $\Rightarrow c = 0$.

$$Thus\ we\ get, x = \frac{k}{2}(2\varphi - sin2\varphi) \quad (13.1.6)$$

$$Also, y = ksin^2\varphi = \frac{k}{2}(1 - cos2\varphi) \quad (13.1.7)$$

If we put $a = \frac{k}{2}$ and $\theta = 2\varphi$ in (13.1.6) and (13.1.7)

Then we have $x = a(\theta - sin\theta)$ and $y = a(1 - cos\theta)$.

We know that these are the parametric equations of the Cycloid.

$\therefore$ The required curve is Cycloid.

13.2 FALLING BODIES

This lesson includes the dynamical problem of a particle falling vertically either freely under the influence of gravity alone or with air resiance.

Newton's second law states that the motion of the acceleration and the mass m and the total force F acting on a particle are related by the relation $F = ma$.

Free Fall is a particle of mass m. Which falls freely under the influence of gravity alone. In this case the only force acting on the particle is mg. Where g is the acceleration due to gravity. If s is the distance travelled at time t then the acceleration is $\frac{d^2s}{dt^2}$.

$$\therefore m\frac{d^2s}{dt^2} = mg \Rightarrow \frac{d^2s}{dt^2} = g$$

$$Integrating\ w.r.to\ t\ we\ get\ velocity\ v = \frac{ds}{dt} = gt + c_1. \quad (13.2.1)$$

Clearly, the constant c_1 is the value of v when $t = 0$.

$\therefore v = gt + v_0$ is the initial velocity.

$$v = gt + \mathrm{v}_0$$

$$Integrating\ (13.2.1)\ w.r.to\ t'we\ get\ velocity$$

$$s = \frac{1}{2}gt^2 + v_0t + c_2. \quad (13.2.2)$$

Clearly, the constant c_2 is the value of s when $t = 0$. say it s_0

$$s = \frac{1}{2}gt^2 + v_0t + s_0 \quad (13.2.3)$$

(13.2.3) is the general solution of (13.2.1).

Corollary:

If the body falls from rest starting at $s = 0$ we have $v_0 = s_0 = 0$.

$\therefore$ by (13.2.2) and (13.2.3) reduce to $v = gt$ and $s = \frac{1}{2}gt^2$.

Eliminating t between these two equations we get $v = \sqrt{2gs}$. Which gives the velocity in terms of the distance travelled.

13.2.1 Retarded Fall

Assume the that air exerts a resisting force proportional to the velocity of the falling body.

The total force acting on the body is given by $mg - k\frac{ds}{dt}$.

The differential equation of the motion is given by

$$m\frac{d^2s}{dt^2} = mg - k\frac{ds}{dt}$$

$$\Rightarrow \frac{d^2s}{dt^2} = (g - c)\frac{ds}{dt},\ \ where\ \ c = \frac{k}{m}.$$

Now, put

$$\frac{ds}{dt} = v. \Rightarrow \frac{dv}{dt} = \frac{d^2}{dt^2} = g - cv.$$

$$\therefore \frac{dv}{g-cv} = dt.$$

$$Hence, -(\frac{1}{c})log(g - cv) = t + logc_1.$$

$$(ie.,) \quad log(g - cv) = -ct - clogc_1.$$

$$(ie.,) \quad log(g - cv){c_1}^c = -ct. \Rightarrow (g - cv) = c_2e^{-ct}.$$

Assume the initial condition $v = 0$ when $t = 0$. We get $c_2 = g$.

$$g - cv = ge^{-ct} \Rightarrow \therefore v = \frac{g}{c}(1 - e^{-ct}).$$

If $t \to \infty, v \to \frac{g}{c}$ then the limiting value of v is $\frac{g}{c}$ is called the **Terminal velocity**.

UNIT-XIV TAUTOCHRONOUS PROPERTY OF THE CYCLOID

Structure

14.1 ORTHOGONAL TRAJECTORIES

Consider a one parameter family of curves given by the equation $f(x,y,c) = 0$ where c is a parameter. A family of curves that intersects each curve of the given family orthogonally is called the family of **orthogonal trajectories** for the given family.

Now, we described the method of finding orthogonal trajectories.

Let $f(x,y,c) = 0$ be a one parameter family of curves. It's differential equation(DE) is $\frac{dy}{dx} = F(x,y)$. These curves can be characterized by the fact that at any point (x,y) on any one of the curves the slope is given by $F(x,y)$.

$\therefore$ The orthogonal trajectory is $-(\frac{dx}{dy}) = F(x,y)$.

The solution of this differential equation gives the required orthogonal trajectories. If the curve in **Polar Co-ordinates** then the angle Ψ which makes the tangent with the initial line is given by the formula $tan\Psi = r(\frac{d\theta}{dr})$.

To find the orthogonal trajectory we replace $r(\frac{d\theta}{dr})$ by $-\frac{1}{r}(\frac{dr}{d\theta})$ to obtain the differential equation of the orthogonal trajectories.

14.2 Procedure for finding an Orthogonal Trajectories

14.2.1 If the curves are in Cartesian Coordinates

1. Obtain the differential equation of the given family of curves. Which can be obtained by differentiating the given equation and eliminating the parameter.
2. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the differential equation obtained in (1) also obtained the differential equation of the orthogonal trajectories.
3. Obtain the general solution of the differential equation obtained

in (2) which is the required family of orthogonal trajectories.

14.2.2 If the curves are in Polar Coordinates

1. Obtain the differential equation of the given family of curves. Which can be obtained by differentiating the given equation and eliminating the parameter.
2. Replace $r\frac{d\theta}{dr}$ by $-\frac{1}{r}(\frac{dr}{d\theta})$ in the differential equation obtained in (1) also obtained the differential equation of the orthogonal trajectories.
3. Obtain the general solution of the differential equation obtained in (2) which is the required family of orthogonal trajectories.

14.3 Examples

14.3.1 Problem

Find the orthogonal trajectories of the family of circles $x^2 + y^2 = a^2$

Solution : Differentiating w.r.to x by $x^2 + y^2 = a^2$ we get $2x + 2yy' = 0$.

(ie.,) $\frac{dy}{dx} = \frac{x}{y}$ which is the differential equation of the given family of circles.

Hence, the differential equation of the orthogonal trajectories is given by

$$-(\frac{dx}{dy}) = -\frac{x}{y} \Rightarrow \frac{dx}{x} = \frac{dy}{y}.$$

$$\therefore logx + logm = logy$$

where m is a constant.

$$\therefore y = mx.$$

Hence the orthogonal trajectories of the given family of circles is the family of straight lines passing through the origin.

14.3.2 Problem

Show that the family of parabolas $y^2 = 4c(x + c)$ is **self orthogonal** in the sense that when a curve in the family intersects another curve of the family then it is orthogonal to it.

Solution :

$$y^2 = 4c(x + c) \qquad (14.3.1)$$

$$yy' = 2c \qquad (14.3.2)$$

Eliminating c from (14.1) and (14.2) we get

$$y = y'(2x + yy') \qquad (14.3.3)$$

(14.3.1) is the differential equation of the given family of parabolas.

$\therefore$ The differential equation of the orthogonal trajectories is given by

$$y = -\frac{1}{y'}[2x + y(-\frac{1}{y'})]$$
$$(ie.,) y(y')^2 = -2xy' + y.$$

$$\Rightarrow y = y'(2x + yy') \qquad (14.3.4)$$

From (14.3.3) and (14.3.4) we see that the differential equation of the given family and the differential equation of the orthogonal trajectories are identical.

Hence, the given family of curves is itself **self orthogonal**.

14.3.3 Problem

Find the orthogonal trajectories of the family of curves given by $r = asin\theta$

Solution :

$$r = asin\theta \Rightarrow \frac{dr}{d\theta} = acos\theta \qquad (14.3.5)$$

Eliminating a from (14.5) we get, $r\frac{d\theta}{dr} = tan\theta$.

The differential equation of the orthogonal trajectories is

$$-\frac{1}{r}(\frac{dr}{d\theta}) = tan\theta$$
$$\Rightarrow -\frac{dr}{r} = tan\theta d\theta$$
$$\Rightarrow -logr = logsec\theta + logc_1$$
$$\Rightarrow logr = logcos\theta + logc$$
$$\Rightarrow r =$$
$ccos\theta \text{ is the equation of the orthogonal trajectories.}$

14.4 Excercises

1. $xy = c$
2. $y = cx^n$
3. $y = ce^x$
4. $r = e^{c\theta}$
5. $r = ccos\theta$.

14.5 TAUTOCHRONOUS PROPERTY IN CYCLOID

Theorem: If a bead is realized at the origin and slides down the wire which is in the form of the cycloid then show that

(i) The time taken to reach the point $(a\pi, 2a)$ at the vertex is given by $\pi\sqrt{\frac{a}{g}}$.

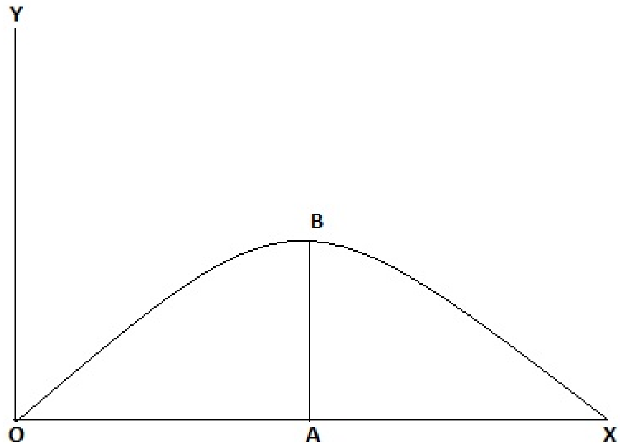
(ii) If at whatever point of the curve the particle starts, the time taken o reach the bottom (vertex) remains the same.

Proof : We know that the equation

$$x = a(\theta - sin\theta) \& y = a(1 - cos\theta)$$

Since, the bead is allowed to slide down the wire which in the form of the cycloid in the Figure 2. We can assume this cycloid is inverted about the $x - axis$.

Figure 2: Geometrical view of Tautochronous



Now, the velocity of the bead at any time is given by

$$v = \frac{dx}{dt} = \frac{\sqrt{dx^2+dy^2}}{dt}$$

$$\Rightarrow v = \frac{\sqrt{[a^2(1-cos\theta)^2+a^2 sin\theta^2]}d\theta}{dt}$$

$$v = \sqrt{2}a\sqrt{(1 - cos\theta)}(\frac{d\theta}{dt}) \qquad (14.3.6)$$

Now, at $\theta = 0; v = 0; y = 0$

$$\therefore kinetic\ energy(K.E) = T = \frac{1}{2}mv^2 = 0$$

$$\therefore potential\ energy(P.E) = V = -mgy = 0$$

At a time t, $t = \frac{1}{2}mv^2$ and $V = -mgy$.

Hence by the principle of conservation energy $\frac{1}{2}mv^2 - mgy = 0$

$$\therefore v^2 = 2gy. \Rightarrow v = \sqrt{2gy} = \sqrt{2ga(1 - cos\theta)}$$

$$\therefore v = \sqrt{2ga}\sqrt{1 - cos\theta} \qquad (14.3.7)$$

Comparing (14.3.6) and (14.3.7) we get,

$$\sqrt{2ga}\sqrt{1 - cos\theta}(\frac{d\theta}{dt}) = \sqrt{2ga}\sqrt{1 - cos\theta}$$

$$\therefore dt = \sqrt{\frac{a}{g}}d\theta \qquad (14.3.8)$$

The time taken to reach the vertex $(a\pi, 2a)$ is given by

$$t = \int_0^{\pi} \sqrt{\frac{a}{g}} d\theta$$

$$\therefore t = \pi\sqrt{\frac{a}{g}} \qquad (14.3.9)$$

Now, suppose the particle starts sliding from y_0 corresponding to $\theta = \alpha$.

By principle of conservation of energy $v = \sqrt{2g(y - y_0)}$

$$\therefore v = \sqrt{2g[a(1 - cos\theta) - a(1 - cos\alpha)]}$$

$$v = \sqrt{2ga(cos\alpha - cos\theta)} \qquad (14.3.10)$$

Comparing (14.3.7) and (14.3.10) we get,

$$dt = \sqrt{\frac{a}{g}(\frac{1-cos\theta}{cos\alpha - cos\theta})} d\theta$$

$\therefore$ The time taken to reach the vertex at $\theta = \pi$ from $\theta = \alpha$ is given by

$$t = \int_{\alpha}^{\pi} \sqrt{\frac{a}{g}(\frac{1-cos\theta}{cos\alpha - cos\theta})} d\theta$$

$$t = \sqrt{\frac{a}{g}} \int_{\alpha}^{\pi} \sqrt{\frac{2sin^2(\frac{\theta}{2})}{(2cos^2(\frac{\alpha}{2})-1)-(2cos^2(\frac{\theta}{2})-1)}} d\theta$$

$$t = \sqrt{\frac{a}{g}} \int_{\alpha}^{\pi} \frac{sin(\frac{\theta}{2})}{\sqrt{cos^2(\frac{\theta}{2})-cos^2(\frac{\theta}{2})}} d\theta$$

$$Put\ cos\frac{\theta}{2} = cos\frac{\alpha}{2}sin\varphi\ \ so\ that,\ \ sin\frac{\theta}{2}d\theta = 2cos\frac{\alpha}{2}cos\varphi d\varphi$$

$$t = -2\sqrt{\frac{a}{g}} \int_{\frac{\pi}{2}}^{0} \frac{cos(\frac{\alpha}{2})d\varphi}{\sqrt{cos^2(\frac{\alpha}{2})-cos^2(\frac{\alpha}{2})sin^2\varphi}}$$

$$t = 2\sqrt{\frac{a}{g}} \int_0^{\frac{\pi}{2}} d\varphi \Rightarrow 2\sqrt{\frac{a}{g}}(\frac{\pi}{2})$$

$$\therefore t = \pi\sqrt{\frac{a}{g}} \qquad (14.3.11)$$

From (14.3.8) and (14.3.11) we notice that the time taken for the bead to slide from the origin to the bottom of the cycloid(vertex) is same as the time taken to the slide from any position of the cycloid to the vertex. This phenomenon is known as **Tautochronous property of the Cycloid**.

MODEL QUESTION PAPER

B.Sc., DEGREE EXAMINATION, NOVEMBER 2019

Mathematics

DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

(2018-2019 onwards)

Time: 3 hours **Maximum: 75 Marks**

PART A **(10×2=20)**

Answer **all** questions.

1. Define exact differential equation.
2. Solve $y = (x - a)p - p^2$ by using clairaut's form.
3. Write the system of simultaneous linear differential equations.
4. Solve $y_2 - 4xy_1 + (4x^2 - 3)y = e^{x^2}$ by using the removal of first derivative.
5. Write the rules for integrating $Pdx + Qdy + Rdz = 0$?
6. Write the classification of integral ?
7. Eliminate a and b from $z = (x + a)(y + b)$.
8. Define the Brachistochrone problem.
9. Write the Auxillary Equation of Charpit's method.
10. Define Self Orthogonal.

PART B **(5×5=25)**

Answer **all** questions choosing either (a) or (b).

11. (a) Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$..

(Or)

(b) Solve $(D^2 + D + 1)^2 y = 0$.

12. (a) Solve $x\frac{d^2y}{dx^2} - (2x - 1)\frac{dy}{dx} + (x - 1)y = e^x$.

(Or)

(b) Solve $z^2dx + (z^2 - 2yz)dy + (2y^2 - yz - xz)dz = 0$.

13. (a) Solve $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$.

(Or)

(b) Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$.

14. (a) Solve: $(i) q = xp + p^2$. $(ii) p = y^2 q^2$

(Or)

(b) Solve $(5+2x)^2 \frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 6x.$

15. (a) Discussed the impact of Retarded Fall in a Falling Bodies.

(Or)

(b) Solve : $q = (z + px)^2$

PART C **(3×10=30)**

Answer any **three** questions.

16. Solve $p^2 + \left(x + y - \frac{2y}{x}\right)p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0.$

17. Solve $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - x + \sin t = 0$; $\frac{d^2y}{dt^2} - \frac{dy}{dt} - y + \cos t = 0.$

18. Describe the necessary and sufficient condition of integrability of $Pdx + Qdy + Rdz = 0.$

19. Analyze the concepts of Tautochronuos property in the Cycloid.

20. Solve the equations $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$, by the method of variation of parameters.

****

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